

# A MODULI SCHEME OF EMBEDDED CURVE SINGULARITIES

JUAN ELIAS \*

## CONTENTS

Introduction	1
1. Truncations of curve singularities.	3
2. Families of embedded curve singularities.	8
3. Moduli space of curve singularities.	12
4. Local properties of the moduli space.	26
References	31

## INTRODUCTION

A central problem in Algebraic Geometry is the classification of several isomorphism classes of objects by considering their deformations and studying the naturally related moduli problems, see [33], [34]. This general strategy has also been applied to singularities. Some classes of singularities with fixed numerical invariants are studied from the moduli point of view, i.e. proving the existence of moduli spaces or giving obstructions to their existence. See for instance [17], [28], [29] and [42].

The main purpose of this paper is to prove the existence of the moduli space  $\mathbf{H}_{N,p}$  parameterizing the embedded curve singularities of  $(\mathbf{k}^N, 0)$  with an admissible Hilbert polynomial  $p$  and to study its basic properties. The main difference between the classical projective moduli problems and the case studied here is that  $\mathbf{H}_{N,p}$  is not a locally finite type scheme. Hence the general techniques of construction of moduli spaces of projective objects do not apply to our problem and we need to develop specific ones. Since  $\mathbf{H}_{N,p}$  is a projective limit of  $\mathbf{k}$ -schemes of finite type we define a measure  $\mu_p$  in  $\mathbf{H}_{N,p}$  valued in the completion  $\widehat{\mathcal{M}}$  of the ring  $\mathcal{M} = K_0(\mathbf{Sch})[\mathbb{L}^{-1}]$  where  $\mathbb{L}$  is the class of  $K_0(\mathbf{Sch})$  defined by the affine line over  $\mathbf{k}$ . This measure induces a motivic integration on  $\mathbf{H}_{N,p}$  and enable us to consider a motivic volume for singularities of arbitrary dimension. See [27], [7], and [30] for the motivic integration on jet schemes.

In [11], see also [14], we characterized the Hilbert-Samuel polynomials of curve singularities: we proved that there exists a curve singularity  $C$  with embedding dimension  $b$  and Hilbert polynomial  $p = e_0 T - e_1$  if, and only if, either  $b = e_0 = 1$  and  $e_1 = 0$ , or  $2 \leq b \leq e_0$ , and  $\rho_{0,b,e_0} \leq e_1 \leq \rho_{1,b,e_0}$ , see Theorem 3.1 for the definitions of  $\rho_{0,b,e_0}$  and  $\rho_{1,b,e_0}$ . Moreover, for each triplet  $(b, e_0, e_1)$  satisfying the above conditions there is a reduced curve singularity  $C \subset (\mathbf{k}^N, 0)$  with  $\mathbf{k}$  an algebraically closed field. From this result and the main result of this

---

*Date:* February 2, 2008.

\*Partially supported by MTM2007-67493.

2000 *Mathematics Subject Classification*, 14H20, 14B07, 14D22.

paper, Theorem 3.8, we deduce that the moduli space  $\mathbf{H}_{N,p}$  is nonempty for the polynomials  $p = e_0 T - e_1$  with  $\rho_{0,b,e_0} \leq e_1 \leq \rho_{1,b,e_0}$  for some  $b \leq N$ .

The contents of the paper is the following. The main purpose of Section 1 is to characterize the zero dimensional closed subschemes  $Z \subset (\mathbf{k}^N, 0)$  for which there exists a curve singularity  $C \subset (\mathbf{k}^N, 0)$  such that  $Z$  is a truncation of  $C$ . In other words we characterize which zero-dimensional schemes can be lifted to a curve singularity, Theorem 1.9. The key idea in the proof of Theorem 1.9 is the control of the dimension of some lifting of  $Z$  obtained by applying Artin's approximation theorem to the system of equations defined by some syzygy conditions deduced from Robbiano-Valla's characterization of standard basis.

It is well known that some properties  $\mathcal{P}$  defined in the set of curve singularities are finitely determined, i.e. determined by the  $n$ -th truncation  $C_n$  of  $C$  for  $n \geq n_0 = n_0(\mathcal{P})$ . The most studied finitely determined property is the analytic type. In [10] we prove that analytic type is finitely determined for  $n \geq n_0 = 2\mu + 1$ , where  $\mu$  stands for the Milnor number of  $C$ , as a corollary we get that if a property of curve singularities is invariant by analytic transformations then is finitely determined. In [10] we also prove that "to have the same tangent cone" or "to have the same Hilbert function" are finitely determined properties. In the section 3 we attach to any finitely determined property  $\mathcal{P}$  a rational power series  $MPS_{\mathcal{P}} \in \mathcal{M}[T]_{loc}$ , Proposition 3.22.

In the second section we introduce the algebraic families of curve singularities over a scheme  $S$ . Notice that the concept of family is a key ingredient in a moduli problem. We analyze the relationship between families and normally flat morphisms and we also give several explicit examples of families of curve singularities with fixed Hilbert polynomial.

The purpose of Section 3 is to construct a moduli scheme  $\mathbf{H}_{N,p}$  parameterizing the embedded curve singularities of  $(\mathbf{k}^N, 0)$  with fixed Hilbert polynomial  $p$ . In the main result of this section, Theorem 3.8, we establish the existence of a  $\mathbf{k}$ -scheme  $\mathbf{H}_{N,p}$  pro-representing the functor of families  $\underline{\mathbf{H}}_{N,p}$ . We will obtain  $\mathbf{H}_{N,p}$  as an inverse limit of  $\mathbf{k}$ -schemes  $\Xi_n$  of finite type with affine morphisms  $a_n : \Xi_n \rightarrow \Xi_{n-1}$ . Notice that in Proposition 3.6 we prove some properties of  $\Xi_n$  as a corollary of Theorem 1.9; in particular we prove that  $\Xi_n$  contains the  $n$ -th truncations of all curve singularities with Hilbert polynomial  $p$ . The key point in the existence of  $\mathbf{H}_{N,p}$  is the control of the behavior of the degree one superficial elements given in [10]; this enables us to prove that  $a_n$  is affine for a big enough  $n$ . As a corollary we get that the cohomological dimension of  $\mathbf{H}_{N,p}$  is finite, and that there exists a universal family over  $\mathbf{H}_{N,p}$ . We end section three by constructing the Hilbert strata of  $\mathbf{H}_{N,p}$  for each admissible Hilbert function, in particular we prove the existence of a moduli space parameterizing normally flat families.

In the second part of section three we introduce a motivic measure  $\mu_p$  defined in the algebra of cylinders  $\mathbf{H}_{N,p}$  valued in  $\mathcal{M}$ . By means of  $\mu_p$  we define for a singularity  $X$  of arbitrary dimension a motivic volume  $vol(X) \in \widehat{\mathcal{M}}$ . Given a property  $\mathcal{P}$  defined on the set of curve singularities with Hilbert polynomial  $p$  we define a motivic Poincare series  $MPS_{\mathcal{P}} \in \mathcal{M}[[T]]$ . We prove that if  $\mathcal{P}$  is a finitely determined property then  $MPS_{\mathcal{P}} \in \mathcal{M}[T]_{loc}$ . In particular we prove  $MPS_{\mathbf{H}_{N,p}} \in \mathcal{M}[T]_{loc}$ .

In the section 4 we compute the tangent space of  $\mathbf{H}_{N,p}$  at a closed point, for this we determine families that are first order deformations. We apply these results to the moduli space of singularities with maximal Hilbert function, in particular to plane curve singularities. By considering  $\mathbf{H}_{N,p}$  as object of the category of pro-schemes we define a topology on its sheaf of rings. Taking dual spaces with respect this topology we obtain the reflexivity of the tangent space at the closed points of  $\mathbf{H}_{N,p}$  and also the reflexivity of the normal space of the curve singularities parameterized by  $\mathbf{H}_{N,p}$ . We end the paper studying the obstructiveness of the closed points of  $\mathbf{H}_{N,p}$ . We prove that plane curve singularities and space curve singularities with maximal numbers of generators with respect to their multiplicity define non-obstructed closed points.

ACKNOWLEDGEMENTS: The author would like to thank O.A. Laudal for the comments and suggestions that improved this paper.

## 1. TRUNCATIONS OF CURVE SINGULARITIES.

Throughout this paper  $\mathbf{k}$  is an algebraically closed field. We set  $R = \mathbf{k}[[X_1, \dots, X_N]]$ ,  $M = (X_1, \dots, X_N)$  is the maximal ideal of  $R$ , and we denote by  $(\mathbf{k}^N, 0)$  the  $\mathbf{k}$ -scheme  $\text{Spec}(R)$ .

A curve singularity of  $(\mathbf{k}^N, 0)$  is a one-dimensional Cohen-Macaulay, closed subscheme  $C$  of  $(\mathbf{k}^N, 0)$ . We denote by  $\mathfrak{m}$  the maximal ideal of  $\mathcal{O}_C = R/I$ , and by  $H_C^1$  (resp.  $h_C^1(T) = e_0(T+1) - e_1$ ) the first Hilbert function (resp. Hilbert polynomial) of  $C$ , i.e.  $H_C^1(t) := \text{length}_R(\mathcal{O}_C/\mathfrak{m}^{t+1})$  and  $H_C^1(t) = h_C^1(t)$  for  $t \geq e_0 - 1$ ;  $e_0$  is the multiplicity of  $C$ . An element  $x \in \mathcal{O}_C$  is a degree one superficial element if  $(\mathfrak{m}^{n+1} : x) = \mathfrak{m}^n$  for all  $n \gg 0$ , see for instance [35].

From now on we fix a degree-one polynomial  $p(T) = e_0(T+1) - e_1$  for which there exist a curve singularity  $C \subset (\mathbf{k}^N, 0)$  of embedding dimension  $b \leq N$  with  $h_C^1 = p$ , see Proposition 3.1.

Given a curve singularity  $C$  we denote by  $C_n$  the closed sub-scheme of  $(\mathbf{k}^N, 0)$  defined by the ideal  $I(C) + M^n$ , we say that  $C_n$  is the  $n$ -th truncation of  $C$ ,  $n \geq 1$ . First we recall some necessary conditions for the ideals  $J \subset R$  to being a truncation of a curve singularity.

**Lemma 1.1.** *Let  $C \subset (\mathbf{k}^N, 0)$  be a curve singularity of multiplicity  $e_0$ . There exists a linear form  $L \in M \setminus M^2$  such that for all  $n \geq e_0 + 1$  such that the following conditions hold:*

- (1)  $\text{length}_R(R/I(C) + M^n + (L)) = e_0$ ,
- (2) *if  $\mathfrak{n}$  is the maximal ideal of  $R/I(C) + M^n$  then for all  $t$ ,  $n-2 \geq t \geq e_0 - 1$ , the product by  $L$  defines an isomorphism of  $\mathbf{k}$ -vector spaces of dimension  $e_0$ :*

$$\frac{\mathfrak{n}^t}{\mathfrak{n}^{t+1}} \xrightarrow{\cdot L} \frac{\mathfrak{n}^{t+1}}{\mathfrak{n}^{t+2}}$$

*Proof.* Since  $\mathbf{k}$  is infinite and  $\mathcal{O}_C$  is a Cohen-Macaulay local ring we may assume that there exists a linear form  $L \in M \setminus M^2$  such that its coset defines a degree one superficial element of  $\mathcal{O}_C$ , [31] Proposition 3.2. Then we have  $\text{length}_R(R/I(C) + (L)) = e_0$  and  $M^n \subset I(C) + (L)$  for all  $n \geq e_0$ . From this we deduce the first equality.

From [26], Theorem 2, we have

$$\dim_{\mathbf{k}} \left( \frac{\mathbf{m}^n}{\mathbf{m}^{n+1}} \right) = e_0$$

for all  $n \geq e_0 - 1$ . Hence from [10], Proposition 1, we deduce

$$\frac{\mathbf{m}^t}{\mathbf{m}^{t+1}} \xrightarrow{\cdot L} \frac{\mathbf{m}^{t+1}}{\mathbf{m}^{t+2}}$$

is an isomorphism of  $\mathbf{k}$ -vector spaces of dimension  $e_0$  for all  $n - 2 \geq t \geq e_0 - 1$ . Since

$$\frac{\mathbf{m}^t}{\mathbf{m}^{t+1}} \cong \frac{\mathbf{n}^t}{\mathbf{n}^{t+1}}$$

for all  $n - 2 \geq t \geq e_0 - 1$ , we get (2).  $\square$

Next we define a set of ideals  $\mathbb{T}_n$  containing the  $n$ -th truncation of curve singularities of multiplicity  $e_0$ . Since we want to consider in Section 3 a scheme structure on some subsets of  $\mathbb{T}_n$ , we replace the identity of Lemma 1.1 (1) by an inequality that will define an open condition on a suitable Grassmanian.

**Definition 1.2.** *Let  $n \geq e_0 + 1$  be an integer,  $\mathbb{T}_n$  is the set of ideals  $J \subset R$  such that  $M^n \subset J$  and such that there exists a linear form  $L \in M \setminus M^2$  such that*

- (1)  $\text{length}_R(R/J + (L)) \leq e_0$ , and
- (2) *if  $\mathbf{n}$  is the maximal ideal of  $R/J$  then the product by  $L$  is an isomorphism of  $\mathbf{k}$ -vector spaces*

$$\frac{\mathbf{n}^t}{\mathbf{n}^{t+1}} \xrightarrow{\cdot L} \frac{\mathbf{n}^{t+1}}{\mathbf{n}^{t+2}}$$

*of dimension  $e_0$ , for all  $t = e_0 - 1, \dots, n - 2$ .*

From the condition (2) it is easy to prove that there exist a linear polynomial  $q_J(T) = e_0(T + 1) - b$ ,  $b \in \mathbb{Z}$ , such that

$$q_J(t) = \text{length}_R(R/J + M^{t+1})$$

for all  $t = e_0 - 2, \dots, n - 1$ . From the characterization of Hilbert functions due to Macaulay, see for instance [40], we get

$$q_J(t) \geq e_0(t + 1) - \binom{e_0}{2}.$$

For all  $n_1 \leq n_2$  we denote by  $a_{n_2, n_1} : \mathbb{T}_{n_2} \longrightarrow \mathbb{T}_{n_1}$  the projection map  $a_{n_2, n_1}(J) = J + M^{n_1}$ .

For all  $f \in R$  we denote by  $f^* \in S = \mathbf{k}[X_1, \dots, X_N]$  the initial form of  $f$ . If  $J$  is an ideal of  $R$  we will denote by  $J^*$  the homogeneous ideal of  $S$  generated by the initial forms of the elements of  $J$ , we put  $\text{Gr}(R/J) = S/J^*$  for the associated graded ring to  $R/J$ . A set of elements of  $J$  such that their initial forms is a (minimal) set of generators of  $J^*$  is known as a (minimal) standard basis of  $J$ . We will denote by  $S_n$ , resp.  $J_n^*$ , the degree  $n$  component of  $S$ , resp.  $J^*$ .

**Proposition 1.3.** *Let  $J$  be an element of  $\mathbb{T}_n$ ,  $n \geq e_0 + 2$ , then every minimal homogeneous basis  $F_1, \dots, F_s$  of  $J^*$  satisfies*

$$\deg(F_i) \notin \{e_0 + 1, \dots, n - 1\}$$

for all  $i = 1, \dots, s$ .

*Proof.* Let  $\mathfrak{n}$  be the maximal ideal of  $R/J$ .  $L$  is a linear form, satisfying the conditions (1) and (2) of the definition of  $\mathbb{T}_n$ , so that

$$\frac{S_t}{J_t^*} = \frac{\mathfrak{n}^t}{\mathfrak{n}^{t+1}} \xrightarrow{.L} \frac{S_{t+1}}{J_{t+1}^*} = \frac{\mathfrak{n}^{t+1}}{\mathfrak{n}^{t+2}}$$

is an isomorphism of  $\mathbf{k}$ -vector spaces for all  $t = e_0 - 1, \dots, n - 2$ . From the surjectivity of these morphisms we get  $S_t = LS_{t-1} + J_t^*$  for  $t = e_0, \dots, n - 1$ . Hence we deduce

$$J_t^* \subset S_1 J_{t-1}^* + LS_{t-1}$$

$t = e_0 + 1, \dots, n - 1$ . Let  $a$  be an element of  $J_t^*$ , then there exist  $b \in S_1 J_{t-1}^* \subset J_t^*$  and  $\alpha \in S_{t-1}$  such that  $a = b + L\alpha$ . In particular  $L\alpha = a - b \in J_t^*$ . From the injectivity of the above morphisms we get  $\alpha \in J_{t-1}^*$ , so  $a \in S_1 J_{t-1}^*$ . Hence  $J_t^* \subset S_1 J_{t-1}^*$  and then

$$J_t^* = S_1 J_{t-1}^*$$

for  $t = e_0 + 1, \dots, n - 1$ . From this we get the claim.  $\square$

**Definition 1.4.** *Let  $J \in \mathbb{T}_n$  be an ideal,  $n \geq e_0 + 2$ , and let  $F_1, \dots, F_s$  be a minimal homogeneous basis of  $J^*$ . We may assume that  $\deg(F_i) \leq e_0$  for  $i = 1, \dots, v$  and  $\deg(F_i) \geq n$  for  $i = v + 1, \dots, s$ . We denote by  $\tilde{J}$  the homogeneous ideal of  $S$  generated by  $F_i$ ,  $i = 1, \dots, v$ .*

Next we will recall a result of G. Hermann, [23], quoted by M. Artin in [1], Theorem 6.5. We need some additional definitions. The degree  $\deg(\mathbf{f})$  of a  $r$ -pla of polynomials  $\mathbf{f} = (f_1, \dots, f_r) \in S^r$  is by definition the sum of the degrees of  $f_1, \dots, f_r$ . The degree of  $\mathbf{F} = \{\mathbf{f}_1, \dots, \mathbf{f}_s\}$ ,  $\mathbf{f}_i \in S^r$ , is the sum of the degrees of  $\mathbf{f}_1, \dots, \mathbf{f}_s$ . Let  $B \subset S^r$  be a  $S$ -submodule, the degree  $\deg(B)$  of  $B$  is the minimum of the degrees of its systems of generators.

**Proposition 1.5** ([23], [1]). *There exists an integer valued function  $\gamma : \mathbb{N}^2 \rightarrow \mathbb{N}$  such that for all ideals  $K \subset S$  of degree  $\leq d$  there exists a primary decomposition of  $K = K_1 \cap \dots \cap K_r$  such that the following integers are bounded by  $\gamma(N, d)$ :*

- (1) *The number  $r$ , and the degree of each primary ideal  $K_i$ .*
- (2) *The degree of the associated prime ideal  $p_i = \text{rad}(K_i)$  and the exponents  $m_i$  such that  $p_i^{m_i} \subset K_i$ ,  $i = 1, \dots, r$ .*

We will apply the last proposition to our setting.

**Definition 1.6.** *Let  $K$  be a height  $N - 1$  homogeneous ideal of  $S$ . Given a minimal primary decomposition  $K = K_1 \cap \dots \cap K_r$  under the conditions (1) and (2) of Proposition 1.5, we can split this decomposition, after a suitable permutation, in two pieces  $K_{\text{emb}} = K_{w+1} \cap \dots \cap K_r$ , such that  $\text{rad}(K_{\text{emb}}) = M$  and  $K_{\text{no-emb}} = K_1 \cap \dots \cap K_w$  is a perfect height  $N - 1$  ideal of  $S$ .*

**Proposition 1.7.** *There exists a function  $\delta : \mathbb{N}^2 \longrightarrow \mathbb{N}$  such that: let  $J$  be an ideal of  $\mathbb{T}_n$ ,  $n \geq \delta(N, e_0)$ , then the following conditions hold:*

- (1)  *$S/\tilde{J}$  is a one-dimensional graded ring of multiplicity  $e_0$ ,  $\deg(\tilde{J}) \leq \delta(N, e_0)$ , and  $\tilde{J} + M^n = J^*$ .*
- (2) *If  $J$  is the ideal defining the  $n$ -th truncation of a curve singularity  $C$  of multiplicity  $e_0$  then  $S/\tilde{J}$  is the associated graded ring to  $\mathcal{O}_C$ , i.e.  $\tilde{J} = I(C)^*$ .*
- (3) *If  $Syz_1(\tilde{J})$  is the first syzygy module of  $F_1, \dots, F_v$  then it holds  $\deg(Syz_1(\tilde{J})) \leq \delta(N, e_0)$ .*

*Proof.* (1) from the definition of  $\tilde{J}$  we get  $\tilde{J} + M^n = J^*$ . Let us assume that  $S/\tilde{J}$  is a zero dimensional ring. Since  $\tilde{J}$  is generated by homogeneous forms of degree  $\leq e_0$  we have

$$\deg(\tilde{J}) \leq e_0 \binom{N + e_0}{N} = d,$$

so from Proposition 1.5 (2) we get that

$$\text{length}_S(S/\tilde{J}) \leq \binom{N - 1 + \gamma(N, d)}{N - 1} = \eta(N, e_0).$$

We define  $\delta(N, e_0) = \eta(N, e_0) + \binom{e_0}{2} + 1$ . Since  $\text{length}_S(S/\tilde{J}) \geq e_0(n) - \binom{e_0}{2}$  then for  $n \geq \delta(N, e_0)$  we get a contradiction, so  $\dim(S/\tilde{J}) \geq 1$ .

Let  $\mathfrak{n}$  be the maximal ideal of  $S/\tilde{J}$ . From the definition of  $\tilde{J}$  and condition (2) of the definition of  $\mathbb{T}_n$  we get that  $\mathfrak{n}^{t+1} = L\mathfrak{n}^t$  for all  $t \geq e_0 - 1$ , so  $\dim(S/\tilde{J}) \leq 1$ . Since  $\dim(S/\tilde{J}) \geq 1$  we have that  $S/\tilde{J}$  is a one-dimensional graded ring of multiplicity less or equal than  $e_0$ . Let  $\tilde{J} = K_{no-emb} \cap K_{emb}$  be a primary decomposition of  $\tilde{J}$  satisfying Proposition 1.5, where  $K_{no-emb}$  is a perfect height  $N - 1$  ideal and  $K_{emb}$  is a  $M$ -primary ideal. Since  $\delta(N, e_0) \geq \gamma(N, d)$  from Proposition 1.5 we have  $M^n \subset K_{emb}$  and then

$$H_{S/\tilde{J}}^1(n) = H_{S/K_{no-emb}}^1(n) + \text{length}_S \left( \frac{K_{no-emb} + K_{emb}}{K_{emb}} \right).$$

From this we deduce that  $H_{S/\tilde{J}}^0(n) = H_{S/K_{no-emb}}^0(n)$  for all  $n \geq \delta(N, e_0)$ . Since  $S/K_{no-emb}$  is a one-dimensional Cohen-Macaulay graded ring of multiplicity  $e_0(S/K_{no-emb}) = e_0(S/\tilde{J}) \leq e_0$  and  $H_{S/\tilde{J}}^0(e_0 - 1) = e_0$ , second condition of the definition of  $\mathbb{T}_n$ , we get that  $S/\tilde{J}$  has multiplicity  $e_0$ .

(2) see [10], Proposition 2; (3) follows from [39]. □

We denote by  $\beta_{Artin} : \mathbb{N}^4 \longrightarrow \mathbb{N}$  the so-called beta function of Artin, see [1], Theorem 6.1.

**Proposition 1.8.** *There exists a numerical function  $\beta : \mathbb{N}^3 \longrightarrow \mathbb{N}$  such that for all ideals  $J$  of  $\mathbb{T}_n$ ,  $n \geq \delta(N, e_0)$ ,*

$$\beta(N, e_0, n) \geq \beta_{Artin}(N, v(r + 1), 2r, n)$$

*with  $v$  the minimal number of generators of  $\tilde{J}$ , and  $r$  the minimal number of generators of  $Syz_1(\tilde{J})$ .*

*Proof.* From the last result we know that  $v \leq \deg(\tilde{J}) \leq \delta(N, e_0)$  and  $r \leq \deg(\text{Syz}_1(\tilde{J})) \leq \delta(N, e_0)$ . From these inequalities we deduce the claim.  $\square$

**Theorem 1.9.** *For all  $n \geq \delta(N, e_0)$  the set of the associated ideals to  $n$ -th truncations of curve singularities of multiplicity  $e_0$  coincides with*

$$\mathbb{T}'_n = a_{\beta(N, e_0, n), n}(\mathbb{T}_{\beta(N, e_0, n)}).$$

Moreover, given  $J \in \mathbb{T}'_n$  there exists a curve singularity  $C$  of multiplicity  $e_0$  such that  $\mathcal{O}_{C_n} = R/J$ , and  $S/\tilde{J}$  is the associated graded ring to  $\mathcal{O}_C$ .

*Proof.* Let us consider an ideal  $J = (f_1, \dots, f_s)$  of  $\mathbb{T}_{\bar{n}}$ ,  $\bar{n} = \beta(N, e_0, n)$ , such that  $F_i = f_i^*$ ,  $i = 1, \dots, s$ , form a minimal basis of  $J^*$ . We may assume that  $\deg(F_i) \leq e_0$  for  $i = 1, \dots, v$  and  $\deg(F_i) \geq \bar{n}$  for  $i = v+1, \dots, s$ , Proposition 3.7. We denote by  $\tilde{J}$  the homogeneous ideal of  $S$  generated by  $F_i$ ,  $i = 1, \dots, v$ .

Let  $Z$  be the first syzygy module of  $f_1, \dots, f_s$ , and let  $Z^*$  be the first syzygy module of  $F_1 = f_1^*, \dots, F_s = f_s^*$ . Let us recall that there exists a map  $\Phi : Z \rightarrow Z^*$ , see proof of [37] Theorem 1.9, such that for all  $(a_1, \dots, a_s) \in Z$  we have  $\Phi(a_1, \dots, a_s) = (b_1, \dots, b_s)$  with  $b_i$  the initial form of  $a_i$  if  $\deg(a_i) = p - d_i$ , with  $d_i = \deg(F_i)$ ,  $p = \min\{\deg(a_i) + d_1, i = 1, \dots, s\}$ , and zero otherwise.

Let  $R^1, \dots, R^r$  be a minimal system of generators of syzygy module of  $F_1, \dots, F_v$ . From [37], Theorem 1.9, there exist elements  $\hat{R}^1, \dots, \hat{R}^r$  of  $Z$  such that  $\Phi(\hat{R}^i) = R^i$ ,  $i = 1, \dots, r$ . Let  $\tilde{R}^i$  the projection on the first  $v$  components of  $\hat{R}^i$ ,  $i = 1, \dots, r$ . Then we have,  $i = 1, \dots, r$ ,

$$\sum_{j=1}^v \tilde{R}_j^i f_j = 0 \mod (X_1, \dots, X_N)^{\bar{n}}.$$

Let us consider the following system of equations attached to these syzygy conditions

$$(Syz) : \begin{cases} \sum_{j=1}^v X_j^i Y_j = 0 \\ i = 1, \dots, r \end{cases}$$

considered in the polynomial ring  $\mathbf{k}[X_1, \dots, X_N; Y_1, \dots, Y_v, X_1^1, \dots, X_v^r]$ .

Since  $\bar{n} = \beta(N, e_0, n) \geq \beta_{\text{Artin}}(N, v(r+1), 2r, n)$ , from the approximation theorem of Artin, [1] Theorem 6.1, there exists a solution of the system of equations  $(Syz)$

$$\sum_{j=1}^v \bar{R}_j^i g_j = 0$$

$i = 1, \dots, r$ , and such that

$$\begin{cases} g_j = f_j \mod M^n & i = 1, \dots, v \\ \bar{R}^i = \tilde{R}^i \mod M^n & i = 1, \dots, r \end{cases}$$

Let us define  $I = (g_1, \dots, g_v)R$ . Next step is to prove that  $g_1, \dots, g_v$  is a standard basis of  $I$ . We will prove it by means of [37] Theorem 1.9. Notice that  $I = J \mod (X_1, \dots, X_N)^n$ ,

$g_i^* = f_i^* = F_i, i = 1, \dots, v$ . Hence  $R^1, \dots, R^r$  is also a minimal system of generators of the first syzygy module of  $\{g_1^*, \dots, g_v^*\}$ . Since  $\Phi(\overline{R}^i) = \Phi(\tilde{R}^i) = R^i, i = 1, \dots, r$ , and  $\overline{R}^1, \dots, \overline{R}^r$  verifies (Syz), from [37] Theorem 1.9, we get that  $\{g_1, \dots, g_v\}$  is a standard basis of  $I$ , i.e.  $I^* = (g_1^*, \dots, g_v^*) = (F_1, \dots, F_v) = \tilde{J}$ . In particular  $R/I$  is a one-dimensional local ring of multiplicity  $e_0$ , Proposition 1.7 (1).

From the condition (1) of the definition of  $\mathbb{T}_{\overline{n}}$  we get that there exists a linear form  $L$  such that  $\dim_{\mathbf{k}}(R/I + (L)) = e_0$ , so  $R/I$  is a one-dimensional Cohen-Macaulay local ring of multiplicity  $e_0$ . If we define  $C = \text{Spec}(R/I)$  then we deduce the claim.  $\square$

## 2. FAMILIES OF EMBEDDED CURVE SINGULARITIES.

For all  $\mathbf{k}$ -scheme of finite type  $S = \text{Spec}(A)$  we will denote by  $(\mathbf{k}^N, 0)_S$  the affine  $\mathbf{k}$ -scheme  $\text{Spec}(A[[X_1, \dots, X_N]])$ . We denote by  $\pi : (\mathbf{k}^N, 0)_S \longrightarrow S$  the morphism of  $\mathbf{k}$ -schemes induced by the natural morphism of  $\mathbf{k}$ -algebras  $A \longrightarrow A[[\underline{X}]] = A[[X_1, \dots, X_N]]$ . Given a closed subscheme  $Z \subset (\mathbf{k}^N, 0)_S$  we will denote by  $Z_n$  the closed sub-scheme of  $(\mathbf{k}^N, 0)_S$  defined by the ideal  $I(Z_n) = I(Z) + (\underline{X})^n$ , for all  $n \geq 1$ .

Let us denote by  $\underline{\mathbf{H}}_{N,p} : \mathbf{Aff} \longrightarrow \mathbf{Set}$  the contravariant functor such that for all affine  $\mathbf{k}$ -scheme of finite type  $S$

$$\underline{\mathbf{H}}_{N,p}(S) = \left\{ \begin{array}{l} \text{closed subschemes } Z \subset (\mathbf{k}^N, 0)_S \text{ such that } \pi : Z \longrightarrow S \\ \text{is flat and} \\ \text{(i) for all } n \geq e_0 + 1 \text{ the morphism } \pi_n : Z_n \longrightarrow S \text{ is flat with} \\ \text{fibers of length } p(n-1), \\ \text{(ii) for all closed points } s \in S \text{ the fiber } Z_s = Z \otimes_S \mathbf{k}(s) \text{ is a} \\ \text{curve singularity of } (\mathbf{k}^N, 0) \text{ with Hilbert polynomial } p. \end{array} \right\}$$

we say that  $\underline{\mathbf{H}}_{N,p}(S)$  is the set of families of curves over  $S$  with Hilbert polynomial  $p$ .

**Proposition 2.1.** (1)  $\underline{\mathbf{H}}_{N,p}(\text{Spec}(\mathbf{k}))$  is the set of curve singularities of  $(\mathbf{k}^N, 0)$  with Hilbert polynomial  $p$ .

(2) Given a scheme  $Z \subset (\mathbf{k}^N, 0)_S$  the following conditions hold

(2.1) if  $Z$  verifies the condition (i) of the definition of family of curve singularities over  $S$  then  $Z$  is flat over  $S$ ,

(2.2) if  $S$  is reduced and if for all closed points  $s \in S$  the fiber  $Z_s$  is a curve singularity with Hilbert polynomial  $p$  then  $Z$  is a family of curves over  $S$ .

*Proof.* (1) From the definition of family of curve singularities it is easy to see that  $\underline{\mathbf{H}}_{N,p}(\text{Spec}(\mathbf{k}))$  is a set of curve singularities with Hilbert polynomial  $p(T)$ . Let  $C$  be a curve singularity of  $(\mathbf{k}^N, 0)$  with Hilbert polynomial  $p(T)$ . From [26], Theorem 2, we have  $h_C^1(n) = H_C^1(n)$  for  $n \geq e_0 - 1$ , so we have  $C \in \underline{\mathbf{H}}_{N,p}(\text{Spec}(\mathbf{k}))$ .



(2.1) follows the main ideas of the proof of [32], Theorem 55. Let  $Z = \text{Spec}(A[[\underline{X}]]/J)$  be a closed subscheme of  $(\mathbf{k}^N, 0)_S$ , with  $S = \text{Spec}(A)$ , such that for all  $n \geq e_0 + 1$  the morphism  $\pi_n : Z_n \longrightarrow S$  is flat with fibers of length  $p(n - 1)$ . We have to prove that the morphism

$$A \longrightarrow B := \frac{A[[\underline{X}]]}{J}$$

is flat. Let  $f : L \longrightarrow P$  be a monomorphism of finitely generated  $A$ -modules, we have to prove that  $f \otimes \text{Id}_B : L \otimes_A B \longrightarrow P \otimes_A B$  is also a monomorphism.

Since the morphism  $A \longrightarrow B_n := \frac{A[[\underline{X}]]}{J+(\underline{X})^n}$  is flat for all  $n \geq e_0 + 1$ , we have that

$$f \otimes \text{Id}_{B_n} : L \otimes_A B_n \longrightarrow P \otimes_A B_n$$

is also a monomorphism,  $n \geq e_0 + 1$ . Hence

$$\varprojlim f \otimes \text{Id}_{B_n} : \varprojlim (L \otimes_A B_n) \longrightarrow \varprojlim (P \otimes_A B_n)$$

is a monomorphism. Since the  $A$ -modules  $L$  and  $P$  are finitely generated it is well known that

$$\varprojlim (L \otimes_A B_n) \cong L \otimes_A B, \quad \varprojlim (P \otimes_A B_n) \cong P \otimes_A B$$

from this we deduce (2.1). The statement (2.2) follows from [26], Theorem 2.  $\square$

**Remark 2.2.** Notice that the condition (i) implies that for all closed points  $s \in S$  the fiber  $Z_s$  is a 1-dimensional closed sub-scheme of  $(\mathbf{k}^N, 0)$  with Hilbert polynomial  $p$ . Hence (ii) can be changed to

(ii)' for all closed points  $s \in S$  the fiber  $Z_s$  is a Cohen-Macaulay scheme.

**Remark 2.3.** There exist flat morphisms that are not families. See Example 4.4 for a first order deformation of a plane curve singularity that is not a family.

In the following examples we will construct families of curve singularities with a closed fiber  $C$  by deforming the ideal  $I(C)$ , by deforming a first syzygy matrix of  $I(C)$  and by deforming a parametrization of  $C$ .

**Example 2.4.** Let  $F, G_1, \dots, G_r$  be power series in the variables  $X_1, X_2$ . We assume that  $\text{order}(F) = e_0$  and  $\text{order}(G_i) \geq e_0 + 1$  for  $i = 1, \dots, r$ . We put  $A = \mathbf{k}[T_1, \dots, T_r]$ ,  $S = \text{Spec}(A)$ ,  $I = (F + \sum_{i=1}^r T_i G_i) \subset A[[X_1, X_2]]$ , and  $Z = \text{Spec}(A[[X_1, X_2]]/I)$ . Notice that for all point  $s$  of  $S$  the fiber  $Z_s$  is a plane singularity of multiplicity  $e_0$ . From Proposition 2.1 (2.2) we deduce that  $Z$  is a family of plane curve singularities with Hilbert polynomial  $p = e_0 T - e_0(e_0 - 1)/2$ .

**Example 2.5.** Let us consider the curve singularity of  $(\mathbf{k}^3, 0)$  defined by the ideal generated by the maximal minors of the matrix ([9])

$$\begin{pmatrix} X_3 & 0 \\ X_1^{e_0-1} & X_3 \\ 0 & X_2 \end{pmatrix}$$

An straightforward computation shows that  $h_C^1 = e_0T - (e_0^2 - 3e_0 + 4)/2$ , and

$$H_C^1 = \{1, 3, 4, 5, 6, \dots, e_0 - 1, e_0, e_0, \dots\}.$$

Let us consider the closed subscheme  $Z$  of  $(\mathbf{k}^N, 0)_S$ ,  $S = \text{Spec}(\mathbf{k}[U])$ , defined by the ideal generated by the maximal minors of the matrix

$$\begin{pmatrix} X_3 + Q_1U & 0 \\ X_1^{e_0-1} + Q_4U & X_3 + Q_2U \\ 0 & X_2 + Q_3U \end{pmatrix}$$

where  $Q_1, Q_2, Q_3$  are formal power series in  $X_1, X_2, X_3$  of order at least 2, and  $Q_4$  is a power series in the same set of variables of order at least  $e_0$ . We know that  $Z$  is a flat deformation of  $C$  with base  $S$ , see [2]. We have a better result: from [37], Theorem 1.9, the tangent cone of the fiber  $Z_s$  coincides with  $Z_0 = C$  for all closed point  $s \in S$ . Hence the Hilbert function is constant on the fibers, from Proposition 2.1 we get that  $Z$  is a family of curve singularities with Hilbert polynomial  $p = e_0T - (e_0^2 - 3e_0 + 4)/2$ .

**Example 2.6.** Notice that in the previous example we have deformed the matrix of syzygies of the curve singularity obtaining a family of curve singularities. We can also deform the parametrization, i.e. the normalization morphism, in order to obtain families. In this case the families verify a stronger condition the singularity order of the fibers is constant, see [41]. Let  $C$  the curve singularity of  $(\mathbf{k}^4, 0)$  with normalization morphism  $\mathcal{O}_C \longrightarrow \overline{\mathcal{O}}_C \cong \mathbf{k}[[t]]$ , defined by

$$X_1 = t^6, X_2 = t^7, X_3 = t^{10}, X_4 = t^{15}.$$

Recall that we can compute the Hilbert function of  $C$  by two different methods. First we can compute the ideal of  $C$  eliminating  $t$  and then computing the Hilbert of  $\mathcal{O}_C$ . In our setting we can also compute the Hilbert function of  $C$  using the fact

$$\delta(C) = \#(\mathbb{N} \setminus \Gamma(C))$$

where  $\Gamma(C) = \langle 6, 7, 10, 15 \rangle$  is the semi-group generated by  $C$ . Hence we have  $\delta(C) = 8$ . On the other hand we can desingularize  $C$  by an unique Blow-up, so from [25] we get that  $\delta(C) = \rho = 8$ . Hence we have  $h_C^1 = 6T - 8$ . Let us consider the family of parameterizations

$$\beta(U) : X_1 = t^6, X_2 = t^7, X_3 = t^{10}, X_4 = t^{15} + Ut^{16}.$$

Then we can consider the closed sub-scheme of  $(\mathbf{k}^4, 0)_S$ ,  $S = \text{Spec}(\mathbf{k}[U])$ , defined by  $\beta(U)$ . From [41] we get that for all closed points  $s \in S$  the normalization of the fiber  $Z_s$  is defined by  $\beta(s)$ . Hence  $\Gamma(Z_s) = \Gamma(C)$  and  $h_{Z_s}^1 = h_C^1 = 6T - 8$ ; from Proposition 2.1 we get that  $Z \in \mathbf{H}_{4,6T-8}(S)$ .

We end this section studying the relationship between families of curve singularities and normally flat morphisms, see [24].

**Definition 2.7.** Let  $S = \text{Spec}(A)$  be a scheme of finite type and let  $Z$  be a closed subscheme of  $(\mathbf{k}^N, 0)_S$  such that there exists a closed section  $\sigma : S \longrightarrow Z$  of  $\pi : Z \longrightarrow S$ . We say that  $Z$  is normally flat along  $S$  if and only if  $\text{Gr}_{\mathcal{I}_{\sigma(S)}}(\mathcal{O}_Z)$  is a flat  $\mathcal{O}_S$  module.

**Proposition 2.8.** Let  $Z$  be a closed subscheme of  $(\mathbf{k}^N, 0)_S$  such that there exists a closed section  $\sigma : S \longrightarrow Z$  of  $\pi : Z \longrightarrow S$ .

- (1) if  $Z$  is a normally flat scheme along  $S$  and verifies (ii)' then  $Z$  is a family of curves,
- (2) if  $Z$  is a family of plane curve singularities then  $Z$  is normally flat along  $S$ .

*Proof.* (1) The result follows from the fact  $\text{Gr}_{\mathcal{I}_{\sigma(S)}}(\mathcal{O}_Z)$  is a flat  $\mathcal{O}_S$ -module if and only if  $\mathcal{O}_Z/\mathcal{I}_{\sigma(S)}^n = \mathcal{O}_{Z_n}$  is a flat  $\mathcal{O}_S$ -module for all  $n \geq 1$ .

(2) Let  $Z$  be a family of plane curve singularities over an affine scheme  $S = \text{Spec}(A)$ . We need to prove that for all closed point  $s \in S$  and  $n \geq 1$  the  $\mathcal{O}_{S,s}$ -module  $\mathcal{O}_{Z_n}$  is free of rank  $e_0 n - e_0(e_0 - 1)/2$ . Let  $m$  be the maximal ideal of  $A$  defined by  $s$ . For all  $n \geq 1$ , we will prove that

$$\frac{A_m[[X_1, X_2]]}{I(Z)A_m[[X_1, X_2]] + (X_1, X_2)^n}$$

is a free  $A_m$ -module of rank  $e_0 n - e_0(e_0 - 1)/2$ . Let  $F \in I(Z)A_m[[X_1, X_2]] \subset A_m[[X_1, X_2]]$  be a power series such that  $A_m[[X_1, X_2]]/(F) \otimes_A \mathbf{k}$  is the local ring  $\mathcal{O}_{Z_s}$ . Since the  $\mathbf{k}$ -vector spaces  $(F) + (X_1, X_2)^{e_0+1} \subset I(Z)A_m[[X_1, X_2]] + (X_1, X_2)^{e_0+1}$  have the same codimension  $e_0(e_0 + 1) - e_0(e_0 - 1)/2$ , the vector spaces agree. From this it is easy to prove (2).  $\square$

**Remark 2.9.** The last Proposition enable us to consider the condition (i) of the definition of family of curve singularities as a weak form of normally flat morphism. Notice that the last three examples are in fact normally flat families.

**Example 2.10.** Example of family not normally flat. Let us consider the family of monomial curves

$$X_1 = t^7, X_2 = t^8, X_3 = (1 - u)t^9 + at^{10}.$$

Since the singularity order and the Hilbert polynomial does not depend on the parameter  $u$ , from [41] and Proposition 2.1, we get that there exists a family of curve singularities  $\pi : Z \longrightarrow S = \text{Spec}(\mathbf{k}[u])$  such that  $Z_u$  is the monomial curve defined by  $u$ . On the other hand  $H_{Z_0}^1(3) = 5$ , and  $H_{Z_1}^1(3) = 6$  so  $\pi$  is not a normally flat family.

In [11] we defined rigid Hilbert polynomials as the polynomials that determines the Hilbert function; i.e.  $p = e_0 T - e_1$  is rigid if there exists a function  $H_p : \mathbb{N} \longrightarrow \mathbb{N}$  such that if  $C$  is a curve singularity with  $h_C^1 = p$  then  $H_C^1 = H_p$ . For instance  $p = e_0 T - e_1$  with  $e_1 = e_0 - 1$ ,  $e_0$ ,  $e_0(e_0 - 1)/2 - 1$ ,  $e_0(e_0 - 1)/2$  are rigid polynomials, and any Hilbert polynomial  $p = e_0 T - e_1$  with  $e_0 \leq 5$  is rigid, see [11]. See also [15] for further results on rigid polynomials. Finally, it is easy to prove

**Proposition 2.11.** Every family of curve singularities with a rigid Hilbert polynomial over a reduced base is normally flat.

### 3. MODULI SPACE OF CURVE SINGULARITIES.

The purpose of this section is to construct a moduli scheme  $\mathbf{H}_{N,p}$  parameterizing the embedded curve singularities of  $(\mathbf{k}^N, 0)$  with fixed Hilbert polynomial  $p$ , Theorem 3.8.

Let us recall that we assumed in the first section that  $p = e_0(T + 1) - e_1$  is an admissible Hilbert polynomial, i.e. there exists a curve singularity  $C$  of  $(\mathbf{k}^N, 0)$  with Hilbert polynomial  $p$ . In [11] we characterized the admissible Hilbert polynomials. To recall that result we have to define some integers attached to  $e_0$  and the embedding dimension: given integers  $1 \leq b \leq e_0$  we consider the following integers  $\rho_{0,b,e_0} = (r+1)e_0 - \binom{r+b}{r}$ , with  $r$  the integer such that  $\binom{b+r-1}{r} \leq e_0 < \binom{b+r}{r+1}$ , and  $\rho_{1,b,e_0} = e_0(e_0 - 1)/2 - (b-1)(b-2)/2$ .

**Proposition 3.1.** *There exists a curve singularity  $C$  with embedding dimension  $b \leq N$  and Hilbert polynomial  $p(T) = e_0T - e_1$  if, and only if, either*

- (1)  $b = 1, e_0 = 1, e_1 = 0$ , or
- (2)  $2 \leq b \leq e_0$ , and  $\rho_{0,b,e_0} \leq e_1 \leq \rho_{1,b,e_0}$ .

*Moreover, for each triplet  $(b, e_0, e_1)$  satisfying the above conditions there is a reduced curve singularity  $C \subset (\mathbf{k}^N, 0)$  with embedding dimension  $b$  and Hilbert polynomial  $p = e_0T - e_1$ , with  $\mathbf{k}$  is an algebraically closed field.*

Let  $F : \mathbb{N} \longrightarrow \mathbb{N}$  be a numerical function such that  $F(t) \leq b(t) := \binom{N+t-1}{N}$  for all  $t \geq 0$ . For each  $t \geq 0$  we denote by  $G_t$  the Grassmannian of  $F(t)$ -dimensional quotients of  $R_t = R/M^t$ , notice that  $R_t$  is a  $b(t)$  dimensional  $\mathbf{k}$ -vector space. Recall that  $G_t$  represents the contravariant functor  $\underline{G}_t : \mathbf{Sch} \longrightarrow \mathbf{Set}$ , where  $\mathbf{Sch}$  is the category of  $\mathbf{k}$ -schemes locally of finite type,  $\mathbf{Set}$  the category of sets, and  $\underline{G}_t(S)$  is the set of locally free quotients of  $\widetilde{R}_{t(S)}$  of rank  $F(t)$ , see [22]-I-9.7.4. If  $K$  is a  $F(t)$ -dimensional quotient of  $R_t$  then we will denote by  $[K]$  the corresponding closed point of  $G_t$ .

We denote by  $F_{r,n}$  the contravariant set-valued functor on  $\mathbf{Sch}$  defined by:  $F_{r,n}(S)$  is the set of  $S$ -module quotients  $\mathcal{F} = \widetilde{R}_{n(S)}/N$  such that the  $\mathcal{O}_S$ -module

$$\mathcal{F}^{(i)} = \widetilde{R}_{n-i(S)}/(\sigma_{n,i})_*(N)$$

belongs to  $\underline{G}_{n-i}(S)$ ,  $i = 0, 1, \dots, n-r$ , where  $\sigma_{n,i} : \widetilde{R}_n \longrightarrow \widetilde{R}_{n-i}$  is the natural morphism of sheaves. For all integers  $r \leq n$ , let  $W(r, n, F)$  be the reduced subscheme of  $G_n$  whose closed points correspond to the  $\mathbf{k}$ -vector space quotients  $R_n/E$  such that  $\dim_{\mathbf{k}}(R_n/E + M^t) = F(t)$  for all  $t = r, \dots, n$ .

**Proposition 3.2.** *The scheme  $W(r, n, F)$  represents the functor  $F_{r,n}$ .*

*Proof.* In order to prove the result we will use [22]-0-4.5.4. From the local nature of the definition of  $F_{r,n}$  it is easy to verify the second condition of [22]-0-4.5.4. The first condition follows from [22]-I-9.7.4.6, and the third condition from [22]-I-9.7.4.7.

We will prove the fourth condition of [22]-0-4.5.4. Let  $\{m_i\}_{i=1, \dots, b(n)}$  be a lexicographically ordered set of monomials of  $\mathbf{k}[X_1, \dots, X_N]$  such that their cosets in  $R_n$  form a  $\mathbf{k}$ -basis. We

denote by  $\mathcal{H}$  the set of  $H = \{i_1, \dots, i_{F(n)}\} \subset \{1, 2, \dots, b(n)\}$  such that

$$\text{card}(H \cap \{1, 2, \dots, b(n-i)\}) = F(n-i)$$

for all  $i = 1, 2, \dots, n-r$ . It is known that for every  $H \in \mathcal{H}$  we get an open set  $B_n(H)$  of  $G_n$ :  $B_n(H)$  is the set of  $F(n)$  dimensional vector spaces  $R_n/L$  such that the cosets of  $m_{i_1}, \dots, m_{i_{F(n)}}$  in this quotient form a  $\mathbf{k}$ -basis. This open set is isomorphic to the affine space of matrices,  $F(n) \times (b(n) - F(n))$ .

Let  $\varphi_H : \mathcal{O}_{\text{Spec}(\mathbf{k})}^{F(n)} \longrightarrow \widetilde{R}_n$  be the morphism of  $\mathcal{O}_{\text{Spec}(\mathbf{k})}$ -modules defined by the  $\mathbf{k}$ -linear map  $\varphi_H : \mathbf{k}^{F(n)} \longrightarrow R_n$  with  $\varphi_H((a_i)_{i \in H}) = \sum_{i \in H} a_i m_i$ . By [22]-0-4.5.4 we get that  $B_n(H)$  is represented by the subfunctor  $\underline{G}_{n,H}$  of  $\underline{G}_n$  defined by:  $\underline{G}_{n,H}(S)$  is the set of  $\mathcal{F} \in \underline{G}_n(S)$  such that the composition

$$\mathcal{O}_S^{F(n)} \xrightarrow{\varphi_H} \widetilde{R}_n(S) \longrightarrow \mathcal{F}$$

is an epimorphism. It is easy to see that there exists a one-to-one correspondence  $\gamma$  between  $\underline{G}_{n,H}(S)$  and the set of  $\mathcal{O}_S$ -morphisms  $v : \widetilde{R}_n \longrightarrow \mathcal{O}_S^{F(n)}$  such that  $v\varphi_H = Id$ ;  $v$  corresponds to  $\mathcal{F} = \widetilde{R}_n / \text{Ker}(v)$ . Hence we have a exact sequence of set maps

$$\underline{G}_{n,H}(S) \xrightarrow{\gamma} \text{Hom}_{\mathcal{O}_S}(\widetilde{R}_n, \mathcal{O}_S^{F(n)}) \xrightleftharpoons[\beta_H]{\alpha_H} \text{Hom}_{\mathcal{O}_S}(\mathcal{O}_S^{F(n)}, \mathcal{O}_S^{F(n)}),$$

with  $\alpha_H(v) = v\varphi_H$ ,  $\beta_H(v) = Id$ ; we get that  $\underline{G}_{n,H}$  is representable by  $B_n(H)$ , i.e. the kernel of the pair of morphisms

$$\mathbf{k}^{b(n)F(n)} \xrightleftharpoons[\beta_H]{\alpha_H} \mathbf{k}^{F(n)^2}.$$

Let us consider the subfunctor  $F_{r,n,H}$  of  $F_{r,n}$  such that for every  $S$ ,  $F_{r,n,H}(S)$  is the set of  $\mathcal{F} \in F_{r,n}(S)$  such that the composition

$$\mathcal{O}_S^{F(n)} \xrightarrow{\varphi_H} \widetilde{R}_{n(S)} \longrightarrow \mathcal{F}$$

is an epimorphism. Let us consider the restriction of  $\gamma$

$$\gamma : F_{r,n,H}(S) \longrightarrow \text{Hom}_{\mathcal{O}_S}(\widetilde{R}_n, \mathcal{O}_S^{F(n)}),$$

first of all we need to compute the image of  $\gamma$ . For this consider the projection in the first  $F(n-i)$  components

$$\pi(i) : \mathcal{O}_S^{F(n)} \longrightarrow \mathcal{O}_S^{F(n-i)},$$

and the canonical monomorphism

$$\sigma(i) : \widetilde{M^{n-i}} / M^n \longrightarrow \widetilde{R}_n.$$

From [22]-0-5.5.7 it is easy to prove

**CLAIM:** Let  $v : \widetilde{R}_{n(S)} \longrightarrow \mathcal{O}_S^{F(n)}$  be a morphism such that  $v\varphi_H = Id$ , then the sheaf  $\widetilde{R}_{n(S)} / \text{Ker}(v)$  is locally free of rank  $F(n-i)$  if and only if  $\pi(i)v\sigma(i) = 0$ .

From the claim we can build up the following exact sequence of map sets

$$F_{r,n,H}(S) \xrightarrow{\gamma} \text{Hom}_{\mathcal{O}_S}(\widetilde{R}_n, \mathcal{O}_S^{F(n)})$$

$$\xrightarrow[\delta_H]{\varepsilon_H} \text{Hom}_{\mathcal{O}_S}(\mathcal{O}_S^{F(n)}, \mathcal{O}_S^{F(n)}) \times \prod_{i=1}^{n-r} \text{Hom}(\widetilde{M^{n-i}/M^n}_{(S)}, \mathcal{O}_S^{F(n-i)}),$$

with  $\delta_H(v) = (Id; 0, \dots, 0)$ , and  $\varepsilon_H(v) = (v\gamma_H; \pi(i)v\sigma(i), i = 1, 2, \dots, n-r)$ . From this and [22]-I-9.4.9 we obtain that  $F_{r,n,H}$  is representable by the Kernel, say  $X_H$ , of the pair of morphisms

$$\mathbf{k}^{b(n)F(n)} \xrightarrow[\delta_H]{\varepsilon_H} \mathbf{k}^{F(n)^2} \times \prod_{i=1}^{n-r} \mathbf{k}^{l(n,i)F(n-i)},$$

$l(n, i) = \dim_{\mathbf{k}}(M^{n-i}/M^n)$ , so we get [22]-0-4.5.4(iv) for the family of functors  $\{F_{r,n,H}\}_{H \in \mathcal{H}}$ . Hence  $F_{r,n}$  is representable by a scheme  $X$ , and  $\{X_H\}_{H \in \mathcal{H}}$ , is an open cover of  $X$ . From the proof of the representability of  $X_H$  we deduce that  $X_H$  is a linear subspace of  $B_n(H)$ . From the definition of  $W(r, n, F)$  and [22]-I-4.2.4(ii) we deduce that  $X = W(r, n, F)$ , so we get that  $W(r, n, F)$  represents the functor  $F_{r,n}$ .  $\square$

We denote by  $\text{Hilb}_n$  the Hilbert scheme parameterizing the closed subschemes of  $\text{Spec}(R_n)$  of length  $F(n)$ ;  $\text{Hilb}_n$  represents the contravariant set-valued functor  $\underline{\text{Hilb}}_n$  on  $\mathbf{Sch}$  for which  $\underline{\text{Hilb}}_n(S)$  is the set of morphisms  $f : Z \subset \text{Spec}(R_n) \times S \rightarrow S$  with fibers of length  $F(n)$ , see [18]. Let

$$a_{n+1} : W(r, n+1, F) \cap \text{Hilb}_{n+1} \rightarrow W(r, n, F) \cap \text{Hilb}_n$$

be the morphism of schemes induced by the functorial morphism

$$a_{n+1} : F_{r,n+1} \times_{\underline{G}_{n+1}} \underline{\text{Hilb}}_{n+1} \rightarrow F_{r,n} \times_{\underline{G}_n} \underline{\text{Hilb}}_n,$$

with  $a_{n+1(S)}(\mathcal{F}) = \mathcal{F}^{(1)}$ .

From now on we assume that  $F(t) = p(t-1) = e_0t - e_1$ , and for all  $n \geq e_0 + 1$  we put  $W(n) = W(e_0 + 1, n, p)$  and  $W'(n) = W(n) \cap \text{Hilb}_n$ .

Let  $C \subset (\mathbf{k}^N, 0)$  be a reduced curve singularity. We will denote by  $\delta(C) = \dim_{\mathbf{k}}(\overline{\mathcal{O}}_C/\mathcal{O}_C)$  the order of singularity of  $C$ , here  $\overline{\mathcal{O}}_C$  is the integral closure of  $\mathcal{O}_C$  on its full ring of fractions. We denote by  $\mu(C)$  the Milnor number of  $C$ ; notice that  $\mu(C) = 2\delta(C) - r(C) + 1$  where  $r(C)$  is the number of branches of  $C$ , [5] Proposition 1.2.1.

**Definition 3.3.** We denote by  $C_n(N, p)$  the set of points of  $G_n$  defined by all truncations  $C_n$  where  $C$  is a curve singularity of Hilbert polynomial  $h_C^1 = p$ .

We will prove that  $C_n(N, p)$  is in fact a constructible set of  $G_n$ , see Proposition 3.6.

**Proposition 3.4.** (1) For all  $n \geq e_0 + 1$  it holds  $C_n(N, p) \subset W'(n)$ .

(2) Let  $C$  be a curve singularity with Hilbert polynomial  $h_C^1 = p$ , then its tangent cone is determined by  $[C_n]$ ,  $n \geq e_0 + 1$ .

(3) If  $C$  is reduced then the analytic type of  $C$  is determined by  $[C_n]$ ,  $n \geq 2\mu(C) + 1$ .

*Proof.* (1) We only need to prove that for all curve singularity of multiplicity  $e_0$  it holds  $[C_n] \in W'(n)$ ,  $n \geq e_0 + 1$ . Notice that the closed points of  $W'(n)$  are the quotients  $R/I$  with  $I \subset R$  ideal such that  $M^n \subset I$  and  $\dim_{\mathbf{k}}(R/M^{n-i} + I) = p(n - i - 1)$  for  $i = 0, \dots, n - e - 1$ . In particular if  $C$  is a curve singularity of  $(\mathbf{k}^N, 0)$  with Hilbert  $p$  then  $R/I(C) + M^n$  defines a closed point of  $W'(n)$  for all  $n \geq e_0 + 1$ , see [26], Theorem 5.

(2) follows from Proposition 1.7. From [10], Theorem 6, we deduce (3).  $\square$

Notice that in the last proposition we proved that  $W'(n)$  contains all  $n$ -th truncations of curve singularities with Hilbert polynomial  $p$ . In order to take account of the Cohen-Macaulayness of the curve singularities we need to shrink  $W'(n)$  by intersecting this scheme with some open Zariski subset of  $G_n$ . For this end, given a linear form  $L \in R$  we denote by  $U_n(L)$  the Zariski open set of  $G_n$  whose closed points are the quotients  $R_n/E$  such that  $\dim_{\mathbf{k}}(R_{e_0+1}/(E, L)) \leq e_0$ , where  $(E, L)$  is the ideal generated by  $E$  and  $L$ .

**Proposition 3.5.** (1) *Let  $I$  be an ideal of  $R$  such that  $A = R/I$  is a one-dimensional local ring of multiplicity  $e_0$ . Let  $L$  be an element of  $R$ , the following conditions hold*

- (1)  *$\dim_{\mathbf{k}}(A/LA) \geq e_0$  and we have equality if and only if  $A$  is Cohen-Macaulay and the coset of  $L$  in  $A$  is a degree-one superficial element.*
- (2) *If  $\dim_{\mathbf{k}}(R/I + M^{e_0+1}) = p(e_0)$  then the following conditions are equivalent:*
  - (a)  *$[R/I + M^{e_0+1}]$  belongs to  $U_{e_0+1}(L)$ ,*
  - (b)  *$A$  is Cohen-Macaulay and the coset of  $L$  in  $A$  is a degree-one superficial element.*
- (3) *There exist linear forms  $L_1, \dots, L_s$  of  $R$ ,  $s = e_0(N - 1) + 1$ , such that for all curve singularity  $C$  of  $(\mathbf{k}^N, 0)$  with Hilbert polynomial  $p$  and  $n \geq e_0 + 1$  it holds  $[C_n] \in W'(n) \cap U_n(L_q)$ .*

*Proof.* (1) Let us assume  $\dim_{\mathbf{k}}(A/LA) \leq \infty$ , i.e.  $LA$  is a  $\mathfrak{m}$ -primary ideal of  $A$ . From [38], Cap. I Proposition 3.4, we get (1). By (1) we deduce that (b) is equivalent  $\dim_{\mathbf{k}}(R/I + (L)) \leq e_0$ . It is easy to see that this inequality equivalent to  $\dim_{\mathbf{k}}(R/I + M^{e_0+1} + (L)) \leq e_0$ , i.e.  $[R/I + M^{e_0+1}]$  belongs to  $U_{e_0+1}(L)$ .

(3) Let  $C$  be a curve singularity  $C$  of  $(\mathbf{k}^N, 0)$  with Hilbert polynomial  $p$ . From [43], Chap. I Proposition 3.2, we deduce that if a linear form  $L$  is a non zero-divisor in  $Gr(\mathcal{O}_C)_{red}$  then  $L$  is a degree one superficial element of  $\mathcal{O}_C$ . Let  $L_1, \dots, L_s$ ,  $s = e_0(N - 1) + 1$  be linear forms such that any subset of  $N - 1$  elements is  $\mathbf{k}$ -independent. From this and the (1) it is easy to deduce the claim.  $\square$

We will define a sub-scheme  $\Xi_n$  of  $Hilb_n$  taking account of the condition (2) of the definition of  $\mathbb{T}_n$ . For this end we have to define some special cells of the Grassmannian  $G_n$ , from a deep study of these cells we will deduce that the morphism  $a_n$  is affine for a big enough  $n$ , Proposition 3.7 (1).

Let  $\mathbf{Mon} = \{m_1, m_2, \dots\}$  be the set of monomials of  $S$  ordered with respect to the degree-lexicographic ordering. Given multi-indexes  $i. = \{i_1, \dots, i_{p(e_0-1)}\} \subset \{1, 2, \dots, b(e_0)\}$ ,  $j. = \{j_1, \dots, j_{e_0}\} \subset \{b(e_0) + 1, \dots, b(e_0 + 1)\}$ , and a linear form  $L_q$ , we define  $D_n(i., j., q)$  as the

linear subspace of  $R_n$  generated by the  $p(n-1)$  linear independent monomials

$$m_{i_1}, \dots, m_{i_{p(e_0-1)}}; L_q^r m_{j_1}, \dots, L_q^r m_{j_{e_0}}, \quad r = 0, \dots, n - e_0 - 1.$$

For all  $n \geq e_0$  we fix a  $\mathbf{k}$ -basis  $V_n(i, j, L)$  of  $R_n$  adding to these elements monomials of suitable degree. We denote by  $B_n(i, j, q)$  the Zariski open subset of  $G_n$  with closed points  $[R_n/E]$  such that the projection  $D_n(i, j, q) \rightarrow R_n/E$  is an isomorphism.

For all triplet  $i, j, q$ , we consider the open sets  $B'_n(i, j, q) = B_n(i, j, q) \cap U(L_q)$ ; we set  $B'_n = \cup_{i,j,q} B'_n(i, j, q)$ . We denote by  $\Xi_n$  the open sub- $\mathbf{k}$ -scheme of  $W'(n)$

$$\Xi_n = W'(n) \cap B'_n$$

and  $\Xi_n(i, j, q) = \Xi_n \cap B'_n(i, j, q)$ . Notice that the morphism  $a_n : W'(n) \rightarrow W'(n-1)$  induces morphisms  $a_n : \Xi_n(i, j, q) \rightarrow \Xi_{n-1}(i, j, q)$ ,  $a_n : \Xi_n \rightarrow \Xi_{n-1}$ , for all  $n \geq e_0 + 1$ .

**Proposition 3.6.** (1) *For all curve singularities  $C$  of  $(\mathbf{k}^N, 0)$  with Hilbert polynomial  $p$  there exists indexes  $i, j, q$  such that  $[C_n] \in \Xi_n(i, j, q)$ , for all  $n \geq e_0 + 1$ .*

(2)  *$C_n(N, p)$  is a constructible set and  $C_n(N, p) \subset \Xi_n$ .*

(3) *For all  $i, j, q$  it holds  $a_n^{-1}(\Xi_{n-1}(i, j, q)_{red}) = \Xi_n(i, j, q)_{red}$ .*

*Proof.* (1) Follows from Proposition 1.1 (2) and Proposition 3.5.

(2) From Theorem 1.9 we get that  $C_n(N, p) = a_{\beta(n, e_0)} \dots a_{n+1}(\Xi_{\beta(n, e_0)})$ , so  $C_n(N, p)$  is a constructible set contained in  $\Xi_n$ .

(3) Let  $x = [R/I + M^n]$  be a closed point of  $\Xi_n$  such that  $a_n(x)$  belongs to  $\Xi_{n-1}(i, j, q)_{red}$ . Let  $\mathfrak{n}$  be the maximal ideal of  $R/I + M^n$ . Since  $a_n(x)$  belongs to  $B_{n-1}(i, j, q)$  we have that  $\dim_{\mathbf{k}}(\mathfrak{n}^{e_0+s}/\mathfrak{n}^{e_0+s+1}) = e_0$  for  $s = 1, \dots, n - e_0 - 1$ , so we have

$$\frac{\mathfrak{n}^{e_0-1}}{\mathfrak{n}^{e_0}} \xrightarrow{.L_q} \frac{\mathfrak{n}^{e_0+s-1}}{\mathfrak{n}^{e_0+s}}$$

is an epimorphism for  $s = 1, \dots, n - e_0 - 1$ . Since  $a_n(x) \in W(n)$  we get that these morphisms are in fact isomorphism, so  $x \in \Xi_n(i, j, q)$ .  $\square$

Let  $X, Y$  be  $\mathbf{k}$ -schemes. Given constructible sets  $A \subset X$ ,  $B \subset Y$  we say that a map  $f : A \rightarrow B$  is a piecewise trivial fibration with fiber a  $\mathbf{k}$ -scheme  $F$  if there exists a finite partition of  $B$  in locally closed subsets  $S \subset Y$  such that  $f : f^{-1}(S) \rightarrow S$  is a fibration of fiber  $F$ , see [6].

The key result in the definition of a local motivic integration in the moduli space  $\mathbf{H}_{N,p}$ , Theorem 3.8, is the following result:

**Proposition 3.7.** (1) *For all  $n \geq e_0 + 4$  the morphism  $a_n : \Xi_n \rightarrow \Xi_{n-1}$  is affine.*

(2) *Given integers  $\nu \geq 2$ ,  $n \geq \delta(N, e_0)$ , there exists a constructible set  $\Sigma_\nu^n \subset C_n(N, p)$  such that: for all curve singularities  $C$  with Hilbert polynomial  $p$ ,  $[R/I(C) + M^n]$  is a closed point of  $\Sigma_\nu^n$  if and only the minimal numbers of generators of  $I(C)$  is  $\nu$ .*

(3) *If  $\nu = N - 1$ , i.e. the case of complete intersection singularities, then we set  $\Sigma_{ci}^n := \Sigma_{N-1}^n$  and the restriction of  $a_n$*

$$a_n : \Sigma_{ci}^n \rightarrow \Sigma_{ci}^{n-1}$$



is an exhaustive piecewise fibration with fiber  $F \cong \mathbf{k}^{(N-1)e_0}$ ,  $n \geq \delta(N, e_0)$ .

*Proof.* (1) We will perform the proof in three steps.

**Step 1** For all  $n \geq e_0 + 4$  the cell  $B_n(i, j, q)$  is isomorphic to an affine space  $\mathbf{k}^{s_n}$ , for some integer  $s_n$ , and  $a_n$  restricts to an affine morphism  $a_n : W(n) \cap B_n(i, j, q) \longrightarrow W(n-1) \cap B_{n-1}(i, j, q)$ .

**Proof:** It is easy to see that considering the base  $V_n(i, j, q)$  that the open set  $B_n(i, j, q)$  is isomorphic to the affine space of  $(b(n) - p(n)) \times b(n)$  matrices, and that  $W(n) \cap B_n(i, j, q)$  is a linear subspace of  $B_n(i, j, q)$ . A general element of  $W(n) \cap B_n(i, j, q)$  can be written as follows

$$U = \left( \begin{array}{|c|c|c|c|} \hline \text{Id} & A & 0 & B \\ \hline 0 & 0 & \text{Id} & C \\ \hline \end{array} \right)$$

Hence for all  $n \geq e_0 + 4$  we get  $B_n(i, j, q) \cong \mathbf{k}^{s_n}$ . From this it is easy to see that  $a_n$  is a linear projection between  $\mathbf{k}^{s_n}$  and  $\mathbf{k}^{s_{n-1}}$  where  $a_n(U)$  is the matrix obtained from  $U$  deleting its last row and the last two columns. Hence we get **Step 1**.

**Step 2** For all  $n \geq e_0 + 4$ ,  $a_n : \Xi_n(i, j, q) \longrightarrow \Xi_{n-1}(i, j, q)$  is an affine morphism.

**Proof:** Let us consider the restriction of  $a_n$

$$a_n : W(n) \cap B_n(i, j, q) \longrightarrow W(n-1) \cap B_{n-1}(i, j, q).$$

Since  $W(r) \cap B'_r(i, j, q)$  is an open subset of  $W(r) \cap B_r(i, j, q)$  for  $r = n, n-1$ , and

$$a_n^{-1}(W(n-1) \cap B'_{n-1}(i, j, q)) = W(n) \cap B'_n(i, j, q),$$

from [22]-I-9.1.2 we deduce that

$$a_n : W(n) \cap B'_n(i, j, q) \longrightarrow W(n-1) \cap B'_{n-1}(i, j, q)$$

is affine. Since  $\Xi_r(i, j, q)$  is a closed subscheme of  $W(r) \cap B'_r(i, j, q)$ , for  $r = n, n-1$ , by [22]-I-9.1.16(i),(v) we obtain **Step 2**.

**Step 3** For all  $n \geq e_0 + 4$  the morphism  $a_n : \Xi_n \longrightarrow \Xi_{n-1}$  is affine.

**Proof:** By **Step 2**, Proposition 3.6 (3), and [22]-I-9.1.18 we get that for all  $n \geq e_0 + 4$  the morphism  $a_n : \Xi_n \longrightarrow \Xi_{n-1}$  is affine.

(2) We denote by  $\nu(B) = \dim_{\mathbf{k}}(B)$  the minimal number of generators of a finitely generated  $R$ -module  $B$ . Let  $G_\nu^n$  be the constructible sub-set of  $\Xi_n$  whose closed points  $[R/J]$  verifies  $\nu(J) = \nu$ . We define  $\Sigma_\nu^n = G_\nu^n \cap C_n(N, p)$ , Proposition 3.6.

Let  $C$  be a curve singularity with Hilbert polynomial  $p$ . From [37] and Proposition 1.7

(3) we get  $I(C) \cap M^n \subset I(C)M$ ,  $n \geq \delta(N, e_0)$ , so we have  $\nu(I(C)) = \nu\left(\frac{I(C)+M^n}{M^n}\right)$  and  $[R/I(C) + M^n]$  is a point of  $\Sigma_\nu^n$  if and only if  $\nu = \nu(I(C))$ . Moreover, let  $f_1, \dots, f_\nu$  be elements  $I(C)$  such that their cosets in  $I(C) + M^n/M^n$  form a minimal system of generators, then  $f_1, \dots, f_\nu$  is minimal system of generators of  $I(C)$ .

(3) We set  $\Sigma_{ci}^n := \Sigma_{N-1}^n$ , i.e.  $\nu = N - 1$ . Let  $x = [R/J + M^n]$  be a closed point of  $\Sigma_{ci}^n$ ,  $J = I(C)$  with  $C$  a curve singularity, and  $y = [R/J + M^{n-1}] = a_n(x)$  its image. We may assume that  $y \in \Xi_{n-1}(i, j, q)_{red}$ , for some set of indexes  $i, j, q$ . Let  $U$  be the associated matrix to  $x$ , see **Step 1**. Let  $\mathbf{f} = f_1, \dots, f_\nu$  be  $\nu$  rows of  $U$  such that their cosets in  $R_{n-1}$  form a minimal system of generators of  $J + M^{n-1}$ , see (2). Notice that  $\mathbf{f}$  is a minimal system of generators of  $J + M^n$  and that all entries of  $U$  are determined by  $\mathbf{f}$ .

If  $x' = [R/J' + M^n]$  is a closed point of  $\Sigma_{ci}^n$  such that  $a_n(x') = a_n(x)$  then  $J'$  admits a minimal system of generators  $\mathbf{g} = g_1, \dots, g_\nu$  such that  $f_i = g_i$  modulo  $M^{n-1}$ . From Proposition 3.6 (3)  $x'$  belongs to  $\Xi_n(i, j, q)$  and then defines a matrix  $U'$  with the same shape of  $U$ . Hence  $\mathbf{f} - \mathbf{g}$  is a set of  $\nu$  homogeneous polynomials of degree  $n$  belonging to  $D_n(i, j, L_q)$ , i.e. the monomials corresponding to the matrix  $B$ . Then we have that  $\Sigma_{ci}^n \subset \Sigma_{ci}^{n-1} \times \mathbf{k}^{(N-1)e_0}$ .

Let  $z$  be a closed point of  $\Sigma_{ci}^{n-1} \times \mathbf{k}^{(N-1)e_0}$  such that  $a_n(z) = x$ . Let  $J_\varepsilon$  be the ideal of  $R$  generated by  $f_1 + \varepsilon_1, \dots, f_\nu + \varepsilon_\nu$ , where  $\varepsilon = \varepsilon_1, \dots, \varepsilon_\nu$  is a set of  $\nu$  homogeneous polynomials of degree  $n$  belonging to  $D_n(i, j, L_q)$ , such that  $z = [R/J_\varepsilon + M^n]$ . From the condition (2) of the definition of  $\mathbb{T}_n$  we deduce that  $\dim(R/J_\varepsilon) \leq 1$ ; since  $\nu = N - 1$  we get that  $\dim(R/J_\varepsilon) = 1$  and then  $C = \text{Spec}(R/J_\varepsilon)$  is a curve singularity. From the definition of  $\mathbb{T}_n$  we deduce that  $C$  is a curve singularity with Hilbert polynomial  $p$ , so  $z \in \Sigma_{ci}^n$ .  $\square$

From Proposition 3.7 and [21]-IV-8.2.3, see also [19] exposé VII, we deduce that the inverse system  $\{\Xi_n, a_n\}_{n \geq e_0+1}$  has a limit  $\mathbf{H}_{N,p}$  that we describe as follows. Since the maps  $a_n$  are affine, we have  $\Xi_n = \text{Spec}(\mathcal{A}_n)$  where  $\mathcal{A}_n$  is a quasi-coherent sheaf of  $\mathbf{k}$ -algebras over  $\Xi_n$ , [20] 1.3.7. Then we define  $\mathbf{H}_{N,p} = \text{Spec}(\mathcal{A})$ , with

$$\mathcal{A} = \varinjlim_n \mathcal{A}_n,$$

see [19] exposé VII. In particular we get that for all point  $x$  of  $\mathbf{H}_{N,p}$  it holds

$$\mathcal{O}_{\mathbf{H}_{N,p},x} \cong \varinjlim_n \pi_n^*(\mathcal{O}_{\Xi_n, \pi_n(x)}),$$

[21], 8.2.12.1. Where we have denoted by

$$\pi_n : \mathbf{H}_{N,p} \longrightarrow \Xi_n,$$

the natural projection,  $n \geq e_0 + 1$ . Given  $i > j \geq e_0 + 1$  we define the affine map  $a_{i,j} : \Xi_i \longrightarrow \Xi_j$  by the composition  $a_{i,j} = a_i a_{i-1} \cdots a_{j+1}$ .

**Theorem 3.8.** *The scheme  $\mathbf{H}_{N,p}$  pro-represents the functor  $\underline{\mathbf{H}}_{N,p}$ .*

*Proof.* We will prove that the functor  $\underline{\mathbf{H}}_{N,p}$  is isomorphic to  $h = \text{Hom}(\cdot, \mathbf{H}_{N,p(T)})$ . Let  $S$  be an object of  $\mathbf{Aff}$  and let  $Z \subset (\mathbf{k}^N, 0)_S$  be a family of curves over  $S$  with Hilbert polynomial  $p$ . Since  $Z_n$  is a flat scheme over  $S$  with fibers of length  $p(n)$ , we have  $\mathcal{O}_{Z_n} \in \underline{\text{Hilb}}_n(S)$ . On the other hand, if  $F_{e_0+1,n}$  is the functor that is represented by  $W(n)$  then  $\mathcal{O}_{Z_n} \in F_{e_0+1,n}(S)$ , and

$$\mathcal{O}_{Z_n} \in (F_{e_0+1,n} \times_{\underline{\mathcal{G}}_n} \underline{\text{Hilb}}_n)(S).$$

Hence by Proposition 3.2 there exists a morphism  $\sigma_n(Z) : S \longrightarrow W'(n)$  for all  $n \geq e_0 + 1$ . Recall that  $B'_n$  is an open subset of  $G_n$ , so  $\sigma_n(Z)$  factorizes through  $X_n = W'(n) \cap B'_n$  if and only if for all closed point  $s \in S$ , we have  $\sigma_n(Z)(s) \in B'_n$ . Since  $\sigma_n(Z)(s) = [(Z_s)_n]$  and  $Z$  is a curve singularity with Hilbert polynomial  $p(T)$  by Proposition 3.6 we get  $\sigma_n(Z)(s) \in B'_n$ . Hence we have a morphism

$$\sigma_n(Z) : S \longrightarrow \Xi_n$$

for all  $n \geq e_0 + 1$ . It is easy to see that for all  $n \geq e_0 + 2$  it holds  $a_n \sigma_n(Z) = \sigma_{n-1}(Z)$  so we have a morphism  $\sigma_*(Z) : S \longrightarrow \mathbf{H}_{N,p(T)}$ . From this we get a functorial morphism

$$\sigma : \underline{\mathbf{H}}_{N,p(T)} \longrightarrow h,$$

sending  $Z \in \underline{\mathbf{H}}_{N,p}(S)$  to  $\sigma_*(Z)$ .

To complete the proof we need to prove that  $\sigma(S)$  is bijective for all  $S$ . The injectivity is straightforward. Let  $g : S = \text{Spec}(A) \longrightarrow \mathbf{H}_{N,p}$  a morphism of  $\mathbf{k}$ -schemes. The morphism

$$g_n = \pi_n g : S = \text{Spec}(A) \longrightarrow \Xi_n \subset \text{Hilb}_n$$

defines an ideal  $J_n \subset A[[\underline{X}]]/(X)^n$ , the compatibility relations  $a_n g_n = g_{n-1}$  give us  $J_n + (\underline{X})^{n-1} = J_{n-1}$  for all  $n \geq e_0 + 2$ . If we write  $J = \bigcap_{n \geq e_0+1} J_n$  then it holds  $J_n = J + (\underline{X})^n$  for all  $n \geq e_0 + 1$ . From this we get  $A[[\underline{X}]]/J$  is isomorphic to the limit of the inverse system defined by  $A[[\underline{X}]]/J_n$ ,  $n \geq e_0 + 1$ .

Let us consider the scheme  $Z \subset (\mathbf{k}^N, 0)_S$  defined by the ideal  $J$ . We will prove that  $Z$  is a family and  $\sigma(Z) = g$ . Since  $I(Z_n) = J_n$  we have that  $Z$  verifies condition (i) of the definition of a family. Let  $s$  be a closed point of  $S$ , we have to prove that  $Z_s$  is a curve singularity with Hilbert polynomial  $p(T)$ . Notice that for all  $n \geq e_0 + 1$  it holds  $(Z_s)_n = (Z_n)_s$  so  $Z_s$  is a one-dimensional sub-scheme of  $(\mathbf{k}^N, 0)$  with Hilbert polynomial  $p$ . From  $\Xi_n \subset B_n$  and Proposition 2.3 we get that  $Z$  is a family. Since  $\sigma(Z) = g$  we obtain the theorem.  $\square$

**Remark 3.9.** Notice that from the last result and Proposition 3.6 we get that  $\pi_n(\mathbf{H}_{N,p}) = C_n(N, p)$  is the constructible set of all  $n$ -truncations of curve singularities with Hilbert polynomial  $p$ .

On the other hand, notice that  $\mathbf{H}_{N,p}$  is not a  $\mathbf{k}$ -scheme locally of finite type. Hence from the previous result we cannot deduce the existence of a universal family. In the next result we will construct a universal family for the scheme  $\mathbf{H}_{N,p}$ .

**Theorem 3.10.** *There exists a  $\mathbf{k}$ -scheme  $\mathbf{Z}_{N,p}$ , limit of an inverse system  $\{U_n, \alpha_n\}_{n \geq e_0+1}$ , and a morphism  $\varphi : \mathbf{Z}_{N,p} \longrightarrow \mathbf{H}_{N,p}$  such that for all families of curve singularities  $f : Z \longrightarrow S$  with Hilbert polynomial  $p$  there exists a unique morphism  $\sigma : S \longrightarrow \mathbf{H}_{N,p}$  such that  $Z_n \cong S \times_{\Xi_n} U_n$ ,  $n \geq e_0 + 1$ .*

*Proof.* Let  $\rho_n : T_n \longrightarrow \text{Hilb}_n$  be the universal family of  $\text{Hilb}_n$ ,  $n \geq e_0 + 1$ . Recall that  $T_n$  is a closed sub-scheme of  $\text{Hilb}_n \times \text{Spec}(R_n)$ , and that  $\rho_n$  is the restriction of the projection to the first component. Hence  $\rho_n$  is an affine morphism.

From the definition of  $\Xi_n$  we have that  $\Xi_n$  is an open subscheme of  $Hilb_n$ . If we denote by  $U_n$  the fibred product

$$U_n = T_n \times_{Hilb_n} \Xi_n$$

we get that the induced morphism  $\rho_n : U_n \longrightarrow \Xi_n$  is affine. Let us consider the following commutative diagram

$$\begin{array}{ccc} U_{n+1} & \xrightarrow{\alpha_n} & U_n \\ \rho_{n+1} \downarrow & & \downarrow \rho_n \\ \Xi_{n+1} & \xrightarrow{a_n} & \Xi_n \end{array}$$

where  $\alpha_n$  is the morphism obtained from the universal family  $T_n \longrightarrow Hilb_n$  and the fact that  $\Xi_n$  is an open subset of  $Hilb_n$ . Since  $a_n \rho_{n+1}$  and  $\rho_n$  are affine we get that  $\alpha_n$  is also affine, [22]-I-9.1.16(v). We define  $\mathbf{Z}_{N,p}$  as the limit of the inverse system  $\{U_n, \alpha_n\}_{n \geq e_0+1}$ . We denote by

$$\sigma : \mathbf{Z}_{N,p} \longrightarrow \mathbf{H}_{N,p}$$

the morphism induced by the morphism of inverse systems  $\{\rho_n\} : \{U_n, \alpha_n\} \longrightarrow \{\Xi_n, a_n\}$ . We leave to the reader the proof of the universal property of  $\sigma : \mathbf{Z}_{N,p} \longrightarrow \mathbf{H}_{N,p}$ .  $\square$

In the following result we will prove that  $\mathbf{H}_{N,p}$  has finite cohomological dimension. Recall that if  $Y$  is a scheme, the cohomological dimension  $cd(Y)$  of  $Y$  is the least integer  $i$  such that  $H^j(Y, \mathcal{F}) = 0$  for all quasi-coherent sheaves  $\mathcal{F}$  and  $j > i$ .

**Proposition 3.11.** *There exists a constant  $g(N)$  such that*

$$cd(\mathbf{H}_{N,p}) \leq g(N)p(e_0 + 3)^{2-2/N}.$$

*Proof.* Let  $\mathcal{F}$  be a quasi-coherent sheaf of  $\mathbf{H}_{N,p}$  and let  $j \geq cd(\Xi_{e_0+3})$ . The inverse system  $\{\Xi_n, a_n\}_{n \geq e_0+1}$  defines a direct system of groups  $\{H^j(\Xi_n, \pi_{n*}(\mathcal{F})), a_{n*}\}_{n \geq e_0+1}$ . From [19], Exp. VII Corollaire 5.10, we have

$$H^j(\mathbf{H}_{N,p}, \mathcal{F}) \cong \varinjlim_n H^j(\Xi_n, \pi_{n*}(\mathcal{F})).$$

Since  $a_n$  is an affine morphism for all  $n \geq e_0 + 4$  we get

$$H^j(\Xi_n, \pi_{n*}(\mathcal{F})) \cong H^j(\Xi_{e_0+3}, \pi_{e_0+3*}(\mathcal{F})) = 0.$$

Hence we have  $cd(\mathbf{H}_{N,p}) \leq cd(\Xi_{e_0+3})$ . Since  $\Xi_{e_0+3}$  is a sub-scheme of  $Hilb_{e_0+3}$  we get

$$cd(\Xi_{e_0+3}) \leq \dim(\Xi_{e_0+3}) \leq \dim(Hilb_{e_0+3}).$$

The claim follows from [4].  $\square$

**Proposition 3.12.** *There exists a one-to-one correspondence between the set of curve singularities with Hilbert polynomial  $p$  and the set of rational points of  $\mathbf{H}_{N,p}$ . Moreover, if the cardinality of  $\mathbf{k}$  is strictly greater than the cardinality of  $\mathbb{N}$  then every closed point of  $\mathbf{H}_{N,p}$  is rational.*

*Proof.* The first part follows from the Proposition 3.8 and [22]-I-3.3.5. By construction we have that  $\mathbf{H}_{N,p}$  can be covered by open affine sets  $\text{Spec}(A)$  where  $A$  is a  $\mathbf{k}$ -algebra countably generated. From [16], Proposition 2.6, we obtain the claim.  $\square$

From now on we assume that the cardinality of the ground field  $\mathbf{k}$  is greater than the cardinality of  $\mathbb{N}$ . Hence we have a one-to-one correspondence between the set of curve singularities with Hilbert polynomial  $p$  and the set of closed points of  $\mathbf{H}_{N,p}$ .

**Definition 3.13.** Let  $x$  be a closed point of  $\mathbf{H}_{N,p}$ , we will denote by  $C_x$  the curve singularity defined by  $x$ , and by  $I_x = I(C_x) \subset R = \mathbf{k}[[X_1, \dots, X_N]]$  the ideal associated to  $C$ . The maximal ideal of  $\mathcal{O}_{C_x} = R/I_x$  is  $\mathfrak{m}_x$ .

Given a function  $F : \mathbb{N} \longrightarrow \mathbb{N}$ , we say that  $F$  is admissible for the polynomial  $p(T)$  if  $F(t) = p(t)$  for  $t \geq e_0 - 1$ . Given an admissible function  $F$  for  $p$  and an integer  $r$ ,  $1 \leq r \leq e_0$ , we define a contravariant set valued functor on  $\mathbf{Aff}$  such that for all  $\mathbf{k}$ -scheme of finite type  $S$  we have

$$\underline{\mathbf{H}}_{N,F,r}(S) = \left\{ \begin{array}{l} Z \in \mathbf{H}_{N,p}(S) \text{ such that for all } n = r, \dots, e_0 + 1 \\ \text{the morphism} \\ \pi : Z_n \longrightarrow S \\ \text{is flat with fibers of length } F(n). \end{array} \right\}$$

Notice that  $\underline{\mathbf{H}}_{N,p} = \underline{\mathbf{H}}_{N,F,e_0+1}$ , and that  $\underline{\mathbf{H}}_{N,F,1}(S)$  is the set of normally flat families of base  $S$ . We denote by  $\mathbf{H}_{N,F,r}$  the  $\mathbf{k}$ -scheme  $\mathbf{H}_{N,F,r} = W(r, e_0 + 1, F) \times_{G_{e_0+1}} \mathbf{H}_{N,p}$ . Notice that  $\mathbf{H}_{N,F,r}$  is a closed sub-scheme of  $\mathbf{H}_{N,p}$ . From the Theorem 3.8 it is easy to prove

**Proposition 3.14.**  $\mathbf{H}_{N,F,r}$  represents the functor  $\underline{\mathbf{H}}_{N,F,r}$ .

Given an admissible function  $F$  for  $p$  there is a family of closed sub-schemes of  $\mathbf{H}_{N,p}$

$$\mathbf{H}_{N,F,1} \subset \mathbf{H}_{N,F,2} \subset \dots \subset \mathbf{H}_{N,F,e_0+1} = \mathbf{H}_{N,p}.$$

We say that  $\mathbf{H}_{N,F,r}$  is the Hilbert stratum of  $\mathbf{H}_{N,p}$  with respect to  $(F, r)$ . Recall that it is an open problem to characterize the admissible Hilbert functions, we only have characterized the asymptotically behavior of Hilbert functions, i.e. Hilbert polynomials Proposition 3.1.

Very few properties of Hilbert functions are known, see for instance [12]. If  $p$  is a rigid polynomial we know that there exists a unique admissible function  $F$  associated to  $p$ , in this case we have  $(\mathbf{H}_{N,F,r})_{red} = (\mathbf{H}_{N,F,r+1})_{red}$ , for all  $r = 1, \dots, e_0$ .

We say that a subset  $D$  of  $\mathbf{H}_{N,p}$  is a cylinder if  $D = \pi_n^{-1}(D_n)$  where  $D_n \subset \pi_n(\mathbf{H}_{N,p})$  is a constructible set of  $\Xi_n$  for some  $n \in \mathbb{N}$ . We denote by  $n(D)$  the least integer  $n$  verifying such a condition. Notice that  $D_n = \pi_n(D)$  for  $n \geq n(D)$ .

In the next result we will prove for  $\mathbf{H}_{N,p}$  some results of [3], Proposition 6.5, 6.6, and [6], Lemma 2.4, proved for the jet schemes.

**Proposition 3.15.** (1) *The collection  $\mathcal{C}_{N,p}$  of cylinders of  $\mathbf{H}_{N,p}$  is a Boolean algebra of sets.*  
 (2) *Given a cylinder  $D$  and a family of cylinders  $\{D_i\}_{i \in \mathbb{N}}$  of  $\mathbf{H}_{N,p}$  such that  $D = \bigcup_{i \in \mathbb{N}} D_i$  there exist a finite set of indexes  $K \subset \mathbb{N}$  such that  $D = \bigcup_{i \in K} D_i$ .*

*Proof.* (1) Since  $\pi_n(\mathbf{H}_{N,p}) = C_n(N, p)$  are constructible sets it is easy to prove (1), Section 3.9.  
 (2) Let us consider the cylindrical sets  $Z_n = D \setminus (D_1 \cup \dots \cup D_n)$ ,  $n \geq 0$ . Since  $Z_1 \supset Z_2 \supset \dots$  we have that the claim is equivalent to  $Z_{i_0} = \emptyset$  for some index  $i_0$ . We can assume that there exists an increasing set of integers  $\{n_i\}_{i \geq 1}$ ,  $n_i \geq n(Z_i)$ , such that  $\pi_{n_i}(Z_i) = T_{n_i}$ , with  $T_{n_i}$  a constructible set of  $\Xi_{n_i}$ , and  $T_{n_{i+1}} \subset a_{n_{i+1}, n_i}^{-1}(T_{n_i})$ ,  $i \geq 1$ . Since  $\bigcap_{i \geq 1} Z_i = \emptyset$  from [21], Proposition 8.3.3, we have that  $Z_{i_0} = \emptyset$  for some index  $i_0$ , and then we get (2).  $\square$

We denote by  $K_0(\mathbf{Sch})$  the Grothendieck ring of  $\mathbf{Sch}$ . Let  $\mathbb{L} = [\mathbf{k}]$  be the coset of  $\mathbf{k}$  in  $K_0(\mathbf{Sch})$ , we set  $\mathcal{M} = K_0(\mathbf{Sch})[\mathbb{L}^{-1}]$ . Let  $\mathcal{F} = \{F^n \mathcal{M}\}_{n \in \mathbb{Z}}$  be Kontsevich's filtration of  $\mathcal{M}$ :  $F^n \mathcal{M}$  is the sub-group of  $\mathcal{M}$  generated by the elements of the form  $[V] \mathbb{L}^{-i}$  for  $i - \dim(V) \geq n$ . We denote by  $\widehat{\mathcal{M}}$  the completion of  $\mathcal{M}$  with respect the filtration  $\mathcal{F}$ .

Let  $C$  be a cylinder of  $\mathbf{H}_{N,p}$ , we say that  $C$  is  $c$ -stable at level  $n \geq n(C)$  if

$$[\pi_{n+1}(C)] = [\pi_n(C)] \mathbb{L}^c \in K_0(\mathbf{Sch}),$$

$C$  is  $c$ -stable if there exists an integer  $n_0 \geq n(C)$  such that  $C$  is  $c$ -stable at level  $n$  for all  $n \geq n_0$ . We denote by  $sn(C)$  the least integer  $n_0$  verifying such a condition.

A cylinder  $C$  is called  $c$ -trivial if there exist an integer  $n_0 \geq n(C)$  such that for all  $n \geq n_0$  the morphism  $a_{n+1} : \pi_{n+1}(C) \longrightarrow \pi_n(C)$  is a piecewise trivial fibration with fiber  $F \cong \mathbf{k}^c$ . We denote by  $tn(C)$  the least integer  $n_0$  verifying such a condition. Notice that  $c$ -trivial implies  $c$ -stable.

**Proposition 3.16.** *Let  $D$  be a  $c$ -trivial cylinder of  $\mathbf{H}_{N,p}$  and let  $C$  be a cylinder. Then  $D \cap C$  is  $c$ -trivial and*

$$\mu_p(C, D) := [\pi_n(C) \cap \pi_n(D)] \mathbb{L}^{-c(n+1)} \in \mathcal{M}$$

*does not depend on  $n$ , provided  $n \geq tn(D), n(C)$ .*

*Proof.* From the definition of  $c$ -trivial cylinder we get that

$$[\pi_{n+1}(C) \cap \pi_{n+1}(D)] \mathbb{L}^{-c(n+2)} = [\pi_n(C) \cap \pi_n(D)] \mathbb{L}^c \mathbb{L}^{-c(n+2)} = [\pi_n(C) \cap \pi_n(D)] \mathbb{L}^{-c(n+1)}.$$

Hence the class  $[\pi_n(C) \cap \pi_n(D)] \mathbb{L}^{-c(n+1)}$  is independent of  $n$ , provided  $n \geq tn(D), n(C)$ .  $\square$

Let us consider the cylinder  $\mathbf{H}_{N,p}^{ci} = \pi_n^{-1}(\Sigma_{ci}^n)$ ,  $n \geq \delta(N, e_0)$ , Proposition 3.7 (3). By the last proposition we may define on  $\mathcal{C}_{N,p}$  a finitely additive measure valued in  $\mathcal{M}$

$$\begin{aligned} \mu_p : \mathcal{C}_{N,p} &\longrightarrow \mathcal{M} \\ C &\longrightarrow \mu_p(C, \mathbf{H}_{N,p}^{ci}) = [\pi_n(C) \cap \Sigma_{ci}^n] \mathbb{L}^{-(n+1)(N-1)e_0} \end{aligned}$$

for a big enough  $n$ .

**Proposition 3.17.** *There exists a finitely additive  $\widehat{\mathcal{M}}$ -valued measure defined by*

$$\begin{aligned} \mu_p : \mathcal{C}_{N,p} &\longrightarrow \widehat{\mathcal{M}} \\ C &\mapsto \lim_{i \rightarrow \infty} \mu_p(C, \mathbf{H}_{N,p}^{c_i}) \end{aligned}$$

*Proof.* Let  $C_1, \dots, C_r$  be a disjoint family of cylinders, then we have to prove

$$\mu_p(C_1 \cup \dots \cup C_r) = \sum_{i=1}^r \mu_p(C_i).$$

Notice that we only need to prove the equality for  $r = 2$ , this case follows from the group structure of  $K_0(\text{Sch})$ .  $\square$

In the next proposition we prove some auxiliary results that we will use later on. Since  $\widehat{\mathcal{M}}$  is complete we may consider the norm  $\|\cdot\| : \widehat{\mathcal{M}} \longrightarrow \mathbb{R}^+$ , with  $\|a\| = 2^{-n}$  where  $n$  is the order of  $a$  with respect the filtration  $\mathcal{F}$  of  $\mathcal{M}$ .

**Proposition 3.18.** (1) *If  $D_1, D_2$  are cylinders of  $\mathbf{H}_{N,p}$  such that  $D_1 \subset D_2$  then*

$$\|\mu_p(D_1)\| \leq \|\mu_p(D_2)\|.$$

(2) *If  $D_1, \dots, D_r$  are cylinders of  $\mathcal{H}$  then*

$$\|\mu_p(D_1 \cup \dots \cup D_r)\| \leq \text{Max}\{\|\mu_p(D_i)\|, i = 1, \dots, r\}.$$

*Proof.* (1) We may assume that  $D_i = \pi_n^{-1}(T_i)$ ,  $i = 1, 2$ , where  $T_1 \subset T_2 \subset \pi_n(\mathbf{H}_{N,p})$  are constructible sets,  $n \gg 0$ . If we set  $r_i(n) = c(n+1) - \dim(T_i \cap \pi_n(\mathbf{H}_{N,p}))$ ,  $i = 1, 2$ , then it is easy to prove  $r_1(n) \leq r_2(n)$  for all  $n \gg 0$ . Hence we deduce

$$\|\mu_p(D_1)\| = \lim_n 2^{-r_1(n)} \leq \lim_n 2^{-r_2(n)} = \|\mu_p(D_2)\|.$$

(2) By induction on  $r$  it is enough to prove the result for  $r = 2$ . Let  $d_i = \dim(T_i \cap \pi_n(\mathbf{H}_{N,p}))$ ,  $i = 1, 2$ , then we have

$$c(n+1) - \dim((T_1 \cup T_2) \cap \pi_n(\mathbf{H}_{N,p})) \geq c(n+1) - \text{Max}\{d_1, d_2\}.$$

From this it is easy to get the claim.  $\square$

A subset  $C$  of  $\mathbf{H}_{N,p}$  is measurable with respect  $\mu_p$  if and only if for every real number  $\epsilon \in \mathbb{R}^+$  there exists a sequence of cylindrical sets  $\{C_i\}_{i \geq 0}$  such that  $C \vee C_0 \subset \bigcup_{i \geq 1} C_i$  and  $\|\mu_p(C_i)\| \leq \epsilon$  for all  $i \geq 1$ , where  $C \vee C_0$  stands for the disjoint union  $C \vee C_0 = C \cup C_0 \setminus C \cap C_0$ . We say that  $C$  is strongly measurable if  $C_0 \subset C$ . See [3], [8] Appendix A.

**Proposition 3.19.** (1) *Any cylindrical set is strongly measurable. The collection of measurable set form a finite algebra of sets  $\mathcal{C}_{N,p}^*$ .*

(2) *If  $C$  is a measurable set of  $\mathbf{H}_{N,p}$  then  $\mu_p(C) := \lim_{\epsilon \rightarrow 0} \mu_p(C_i)$  exists in  $\widehat{\mathcal{M}}$  and is independent of the choice of the sequence  $\{C_i\}_{i \geq 0}$ . Then there exists a finite measure*

$$\begin{aligned} \mu_p : \mathcal{C}_{N,p}^* &\longrightarrow \widehat{\mathcal{M}} \\ C &\mapsto \mu_p(C) \end{aligned}$$

(3) Let  $\{C_i\}_{i \geq 0}$  be a sequence of measurable sets.

(3.1) If  $\lim_{i \rightarrow \infty} \|\mu_p(C_i)\| = 0$  then  $\cup_{i \geq 0} C_i$  is measurable.

(3.2) If the sequence of measurable sets are mutually disjoint and  $C = \cup_{i \geq 0} C_i$  is measurable then  $\sum_{i \geq 0} \mu_p(C_i)$  converges in  $\widehat{\mathcal{M}}$  to  $\mu_p(C)$ .

*Proof.* The proof of (1) and (3) are standard. (2) Let  $\epsilon, \epsilon'$  be positive real numbers, and let  $\{C_i\}_{i \geq 0}, \{C'_i\}_{i \geq 0}$  sequences of cylindrical sets of  $\mathbf{H}_{N,p}$  verifying the conditions of measurable set. Then we have  $C_0 \vee C'_0 \subset \cup_{i \geq 1} C_i \cup \cup_{i \geq 1} C'_i$ , from Proposition 3.15 (1) there exist integers  $r, r'$  such that  $C_0 \vee C'_0 \subset \cup_{i=1}^r C_i \cup \cup_{i=1}^{r'} C'_i$ . From this and Proposition 3.18 (2) we get  $\|\mu_p(C_0 \vee C'_0)\| \leq \text{Max}\{\epsilon, \epsilon'\}$ . Notice that  $\mu_p(C_0) - \mu_p(C'_0) = \mu_p(C_0 \setminus C'_0)$ ; Proposition 3.18 (1) and the last inequality yield  $\|\mu_p(C_0) - \mu_p(C'_0)\| \leq \text{Max}\{\epsilon, \epsilon'\}$ . From this it is easy to get the claim.  $\square$

Let  $C$  be a measurable set of  $\mathbf{H}_{N,p}$  and  $f : C \longrightarrow \mathbb{Z} \cup \{\infty\}$ , we say that  $f$  is exponentially integrable if the fibers of  $f$  are measurable and the motivic integral

$$\int_C \mathbb{L}^{-f} d\mu_p := \sum_{s \geq 0} \mu_p(C \cap f^{-1}(s)) \mathbb{L}^{-s}$$

converges in  $\widehat{\mathcal{M}}$ .

Given a singularity  $X \subset (\mathbf{k}^N, 0)$  of arbitrary dimension, let us consider the function

$$\begin{aligned} \gamma_X : (\mathbf{H}_{N,p})_{\text{rat}} &\longrightarrow \mathbb{N} \cup \{\infty\} \\ C &\longrightarrow (C \cdot X) \end{aligned}$$

where  $(C \cdot X)$  stands for the "false" intersection multiplicity:  $(C \cdot X) = \dim_{\mathbf{k}}(R/I(C) + I(X))$ .

**Proposition 3.20.** *Let  $X \subset (\mathbf{k}^N, 0)$  be a singularity of algebraic variety.*

(i) *For all  $s \in \mathbb{N}$  the set  $\gamma_X^{-1}(s)$  is a cylinder,*

(ii)  *$\gamma_X$  is exponentially integrable.*

*Proof.* (i) Notice that if  $(C \cdot X) = s$  then  $M^s \subset I(C) + I(X)$ . From this fact it is easy to see that  $\gamma_X^{-1}(s)$  is a cylindrical subset of  $\mathbf{H}_{N,p}$ . (ii) We can apply Proposition 3.19, (3.2), in order to prove that  $\gamma_X$  is exponentially integrable.  $\square$

Let  $X \subset (\mathbf{k}^N, 0)$  be a singularity of arbitrary dimension. The motivic volume of  $X$  with respect to  $p$  is the integral

$$\text{vol}_p(X) = \int_{\mathbf{H}_{N,p}} \mathbb{L}^{-\gamma_X} d\mu_p = \sum_{s \geq 0} \mu_p(\gamma_X^{-1}(s)) \mathbb{L}^{-s}$$

Given an integer  $e_0$  we denote by  $\mathcal{H}(e_0)$  the finite set of admissible Hilbert polynomials, see Proposition 3.1.



**Theorem 3.21** (definition of motivic volume). *Let  $X$  be a singularity, then the series*

$$\sum_{e_0 \geq 1} \left( \sum_{p \in \mathcal{H}(e_0)} \text{vol}_p(X) \right) \mathbb{L}^{-e_0}$$

*converges in  $\widehat{\mathcal{M}}$  to the motivic volume  $\text{vol}(X)$  of  $X$ .*

*Proof.* Let us consider the motivic volume of  $X$  with respect  $p$

$$\begin{aligned} \text{vol}_p(X) &= \sum_{s \geq 0} \mu_p(\gamma^{-1}(s)) \mathbb{L}^{-s} \\ &= \sum_{s \geq 0} [\pi_n(\gamma^{-1}(s)) \cap \Sigma_{ci}^n] \mathbb{L}^{-((n+1)(N-1)e_0 + s)} \end{aligned}$$

with  $n \geq s, \delta(N, e_0)$ . From Proposition 3.7 we get that

$$[\pi_n(\gamma^{-1}(s)) \cap \Sigma_{ci}^n] \mathbb{L}^{-((n+1)(N-1)e_0 + s)} \in F^l \mathcal{M}$$

for a non-negative integer  $l$ . From this we get

$$\left( \sum_{p \in \mathcal{H}(e_0)} \text{vol}_p(X) \right) \mathbb{L}^{-e_0} \in F^{e_0} \mathcal{M}$$

and we are done.  $\square$

Let  $\mathcal{P}$  be a property defined in the set of curve singularities with Hilbert polynomial  $p$ . Let  $c(\mathcal{P})$  be the set of rational points of  $\mathbf{H}_{N,p}$  corresponding to curve singularities verifying the property  $\mathcal{P}$ . We say that  $\mathcal{P}$  is finitely determined if there exists an integer  $n_0 = n_0(\mathcal{P})$ , that we may assume  $n_0 \geq \delta(N, e_0) + 1$ , Proposition 3.7, such that for all curve singularities  $C^1, C^2$  with Hilbert polynomial  $p$  and  $C_n^1 = C_n^2, n \geq n_0$ , then  $C^1$  verifies  $\mathcal{P}$  if and only if  $C^2$  verifies  $\mathcal{P}$ . Notice that any  $\mathcal{P}$  analytically invariant property defined in the set of reduced curve singularities is finitely determined, [10].

We say that a finitely determined property  $\mathcal{P}$  is constructible, cfd-property for short, if the set of truncations  $[C_n] \in \Xi_n, n \geq n_0(\mathcal{P})$ , such that  $C$  verifies  $\mathcal{P}$  is a constructible set  $\mathbf{c}_n(\mathcal{P})$  of  $\Xi_n$ . Trivially a cfd-property  $\mathcal{P}$  is determined by the cylindrical set  $c(\mathcal{P}) = \pi_n^{-1} \mathbf{c}_n(\mathcal{P})$ . On the other hand every cylindrical set defines a cfd-property.

Given a property  $\mathcal{P}$  of curve singularities with Hilbert polynomial  $p$  we define its motivic Poincare series by,  $n_0 = n_0(\mathcal{P})$ ,

$$MPS_{\mathcal{P}} = \sum_{n \geq n_0} [\pi_n(c(\mathcal{P}))] T^n \in \mathcal{M}[[T]].$$

We denote by  $\mathcal{M}[T]_{loc}$  the ring of rational power series, i.e. the sub-ring of  $\mathcal{M}[[T]]$  generated by  $\mathcal{M}[T]$  and  $(1 - \mathbb{L}^a T^b)^{-1}, a \in \mathbb{N}, b \in \mathbb{N} \setminus \{0\}$ .

**Proposition 3.22.** *Let  $\mathcal{P}$  be a cfd-property then it holds*

$$MPS_{\mathcal{P}} \in \mathcal{M}[T]_{loc}.$$

*Proof.* If we set  $n_0 = n_0(\mathcal{P})$  then we have

$$\begin{aligned} MPS_{\mathcal{P}} &= \sum_{n \geq n_0} [\pi_n(c(\mathcal{P})) \cap \Sigma_{ci}^n] T^n \\ &= \sum_{n \geq n_0} [\pi_{n_0}(c(\mathcal{P})) \cap \Sigma_{ci}^{n_0}] \mathbb{L}^{(N-1)e_0 n} T^n \\ &= [\pi_{n_0}(c(\mathcal{P})) \cap \Sigma_{ci}^{n_0}] \frac{\mathbb{L}^{(N-1)e_0 n_0} T^{n_0}}{1 - \mathbb{L}^{(N-1)e_0} T} \end{aligned}$$

and we get the claim.  $\square$

Notice that the condition "belongs to  $\mathbf{H}_{N,p}$ " is a cfd-property that we will denote by  $\mathbf{H}_{N,p}$ . A simple computation shows:

$$MPS_{\mathbf{H}_{N,p}} = [\Sigma_{ci}^{n_0}] \frac{\mathbb{L}^{(N-1)e_0 n_0} T^{n_0}}{1 - \mathbb{L}^{(N-1)e_0} T}$$

where  $n_0 = \delta(N, e_0) + 1$ .

#### 4. LOCAL PROPERTIES OF THE MODULI SPACE.

The purpose of this section is to study the local ring  $\mathcal{O}_{\mathbf{H}_{N,p},x}$  where  $x$  is a closed point of  $\mathbf{H}_{N,p}$ . In particular we will compute the tangent space  $T_x = Hom_{\mathbf{k}}(\mathbf{n}_x/\mathbf{n}_x^2, \mathbf{k})$  of  $\mathbf{H}_{N,p}$  at  $x$ , where  $\mathbf{n}_x$  is the maximal ideal of  $\mathcal{O}_{\mathbf{H}_{N,p},x}$ .

We denote by  $\mathbf{Aff}'$  the subcategory of  $\mathbf{Aff}$  of the  $\mathbf{k}$ -schemes  $Spec(A)$  where  $A$  is an Artinian local  $\mathbf{k}$ -algebra. Let  $\underline{H}_{N,p(T)}^x$  be the contravariant functor between  $\mathbf{Aff}'$  and  $\mathbf{Set}$ , such that for any object  $S$  of  $\mathbf{Aff}'$  we have

$$\underline{H}_{N,p(T)}^x(S) = \left\{ \begin{array}{l} Z \in \underline{\mathbf{H}}_{N,p(T)}(S) \text{ such that} \\ Z_s = C_x \\ \text{for the closed point } s \text{ of } S. \end{array} \right\}$$

We will denote by  $D$  the  $\mathbf{k}$ -scheme  $Spec(\mathbf{k}[\varepsilon])$ , let  $p_D$  be the closed point of  $D$ . Notice that the elements of  $\underline{H}_{N,p(T)}^x(D)$  are first embedded deformations of  $C_x$ , but not all, see Example 4.4. It is well known that there exists a bijection between  $T_x$  and the set of morphism  $f : D \rightarrow \mathbf{H}_{N,p}$  such that  $f(p_D) = x$ . By Theorem 3.8 we obtain that there exist a bijection between  $T_x$  and  $\underline{H}_{N,p(T)}^x(D)$ . From Theorem 3.8 it is easy to prove

**Proposition 4.1.** *The functor  $\underline{H}_{N,p(T)}^x$  is pro-represented by  $Spec(\mathcal{O}_{\mathbf{H}_{N,p},x})$ .*

In order to compute  $T_x$  we need to characterize the set  $\underline{H}_{N,p(T)}^x(D)$ . Let  $I$  be an ideal of  $R$ , we consider Hironaka's invariant  $v^*(I)$  associated to  $I$ , see [24] Chapter III, definition 1. Let  $f_1, \dots, f_s$  be a standard basis of  $I$  such that  $order(f_i) = v^i(I)$ . Recall that from [10], Proposition 2, we have that  $v^i(I(C)) \leq e_0$ ,  $i = 1, \dots, s$ , for all curve singularity  $C$  of multiplicity  $e_0$ .

**Proposition 4.2.** *Let  $J = (f_1 + \varepsilon g_1, \dots, f_s + \varepsilon g_s)$  be an ideal of  $\mathbf{k}[\varepsilon][[\underline{X}]]$  defining a first order deformation  $\varphi : Z \rightarrow D$  of a curve singularity  $C$  of multiplicity  $e_0$  defined by the ideal  $I = (f_1, \dots, f_s)$ . Then the following conditions are equivalent:*

- (1)  $Z$  is a family of curve singularities with Hilbert polynomial  $p(T)$ ,
- (2)  $\varphi : Z_{e_0+1} \rightarrow D$  is flat,
- (3) for all  $i = 1, \dots, s$  it holds that  $g_i \in (I + M^{e_0+1} : I + M^{e_0+1-v^i(I)})$ ,

*Proof.* We put  $\nu^i = \nu^i(I)$ . We will use the syzygy flatness criterion, see for instance [2], Corollary page 11. From this result, we deduce that  $E_n = \mathbf{k}[\varepsilon][[\underline{X}]]/(J + M^n)$  is a flat  $\mathbf{k}[\varepsilon]$ -module if and only if for all  $a_1, \dots, a_s \in R$  such that  $\sum_{i=1}^s a_i f_i \in M^n$  there exist  $A_1, \dots, A_s \in R$  such that

$$\sum_{i=1}^s (a_i + \varepsilon A_i)(f_i + \varepsilon g_i) \in M^n \mathbf{k}[\varepsilon][[\underline{X}]].$$

By definition of family of curve singularities we get that (1) implies (2).

(2) implies (3). Let us assume that  $\varphi : Z_{e_0+1} \rightarrow D$  is flat. Let  $a$  be an element of  $M^{e_0+1-v^i}$ , we need to show that for all  $i = 1, \dots, s$  we have  $g_i a \in I + M^{e_0+1}$ . From the flatness of  $E_{e_0+1}$  and  $f_i a \in M^{e_0+1}$ , we deduce that there exists  $A_1, \dots, A_s \in R$  such that  $g_i a + \sum_{i=1}^s A_i f_i \in M^{e_0+1}$ , so  $g_i a \in M^{e_0+1} + I$ .

(3) implies (1). Suppose that for all  $i = 1, \dots, s$  it holds

$$g_i \in (I + M^{e_0+1} : I + M^{e_0+1-v^i}),$$

so for all  $n \geq e_0 + 1$  we get  $g_i \in (I + M^n : I + M^{n-v^i})$ . Let  $a_1, \dots, a_s \in R$  be elements with  $\sum_{i=1}^s a_i f_i \in M^n$ . Since  $f_1, \dots, f_s$  is a standard basis of  $I$ , by [37], Corollary 1.8, there exists  $C_i \in M^{n-v^i}$ ,  $i = 1, \dots, s$ , such that  $\sum_{i=1}^s a_i f_i = \sum_{i=1}^s C_i f_i$ , so  $a_1 - C_1, \dots, a_s - C_s$  is a syzygy of  $f_1, \dots, f_s$ . From the flatness of  $Z$  over  $D$ , we deduce that there exist  $A_1, \dots, A_s$  such that  $\sum_{i=1}^s (a_i - C_i + \varepsilon A_i)(f_i + \varepsilon g_i) = 0$ . From the assumption (2) there exist  $B_1, \dots, B_s \in R$  such that

$$\sum_{i=1}^s C_i g_i - \sum_{i=1}^s B_i f_i \in M^n.$$

Since  $\sum_{i=1}^s a_i f_i \in M^n$ , from the two last equalities it is easy to see that

$$\sum_{i=1}^s (a_i + \varepsilon(A_i - B_i))(f_i + \varepsilon g_i) \in M^n.$$

We have proved that  $E_n$  is a  $D$ -module flat for all  $n \geq e_0 + 1$ , so  $Z$  is a family. □

**Corollary 4.3.** *For all closed point  $x$  of  $\mathbf{H}_{N,p}$ , the morphism*

$$d(\pi_n) : T_{\mathbf{H}_{N,p},x} \longrightarrow T_{\Xi_n, \pi_n(x)}$$

*is surjective,  $n \geq e_0 + 1$ .*

Proposition 4.2 enable us to give an example of a first order deformation that is not a family.

**Example 4.4.** Let us consider the plane curve singularity defined by the equation  $X_1^3 = 0$ . In this case we have  $v_1 = 3$  and  $e_0 = 3$ . Let us consider the first order deformation  $Z$  defined by  $X_1^3 + \varepsilon X_1$ , i.e.  $g_1 = X_1$ . From the last result we get that  $Z_4 \rightarrow D$  is not flat.

It is well known that the first order embedded deformations of  $C_x$  are classified by the normal module

$$N_x = \text{Hom}_{R/I_x}(I_x/I_x^2, R/I_x).$$

If we denote by  $\text{embdef}(C_x)$  the first order embedded deformations of  $C_x$ , we will define a bijective map  $\tau : N_x \rightarrow \text{embdef}(C_x)$ . Let  $g : I_x/I_x^2 \rightarrow R/I_x$  be a morphism of  $R/I_x$ -modules. Then a lifting of  $g$  is a  $R/I_x$ -module morphism  $(\bar{g}) = (\bar{g}_1, \dots, \bar{g}_s) : (R/I_x)^s \rightarrow R/I_x$  such that the following diagram is commutative

$$\begin{array}{ccc} (R/I_x)^s & & \\ (f.) \downarrow & \searrow (\bar{g}.) & \\ I_x/I_x^2 & \xrightarrow{g} & R/I_x \end{array}$$

$\tau(g)$  is the first order deformation defined by the ideal  $J = (f_1 + \varepsilon g_1, \dots, f_s + \varepsilon g_s)$  with  $(g_1, \dots, g_s)$  a lifting of  $g$ .

We denote by  $N'_x$  the set of  $g \in N_x$  for which there exist a lifting  $(g_1, \dots, g_s)$  such that for all  $i = 1, \dots, s$  it holds that  $\bar{g}_i \in (\mathfrak{m}_x^{e_0+1} : \mathfrak{m}_x^{e_0+1-v^i})$ . From last Proposition we deduce

**Proposition 4.5.**  $\tau$  defines a bijection between  $T_x$  and  $N'_x$ .

In a very few cases we can compute  $v^*(I(C))$ ; for example if  $C$  is a curve singularity with maximal Hilbert function then  $v^*(I(C)) = \{e_0, \dots, e_0 + 1\}$ , see [36]. Notice in this case the ring  $\text{Gr}_{\mathfrak{m}}(\mathcal{O}_C)$  is Cohen-Macaulay,  $\mathfrak{m}$  is the maximal ideal of  $\mathcal{O}_C$ . We can prove a more general result without any restriction on  $v^*$ .

**Proposition 4.6.** Let  $C$  be a curve singularity of  $(\mathbf{k}^N, 0)$  of multiplicity  $e_0$  and embedding dimension  $b$ . It holds:

- (1) If the associated graded ring  $\text{Gr}_{\mathfrak{m}}(\mathcal{O}_C)$  is Cohen-Macaulay then  $(\mathfrak{m}^{e_0+1} : \mathfrak{m}^{e_0+1-v^i}) = \mathfrak{m}^{v^i}$  for  $i = 1, \dots, s$ .
- (2) If  $C$  has a maximal Hilbert function and  $e_0 = \binom{b-1+r}{r}$ , for some integer  $r$ , then  $I(C)$  has a standard basis of  $s = \binom{b-1+r}{r+1}$  forms of degree  $r+1$  and the ring  $\text{Gr}_{\mathfrak{m}}(\mathcal{O}_C)$  is Cohen-Macaulay. In particular  $(\mathfrak{m}^{e_0+1} : \mathfrak{m}^{e_0-r}) = \mathfrak{m}^{r+1}$ .

*Proof.* (1) We put  $v^i = v^i(I)$ . Notice that we always have  $\mathfrak{m}^{v^i} \subset (\mathfrak{m}^{e_0+1} : \mathfrak{m}^{e_0+1-v^i})$ . Let  $h$  be an element of  $\mathcal{O}_C$  such that  $h \mathfrak{m}^{e_0+1-v^i} \subset \mathfrak{m}^{e_0+1}$ ,  $i = 1, \dots, s$ .

Suppose that  $h \notin \mathfrak{m}^{v^i}$ ; let  $t$  be the least integer such that  $h \in \mathfrak{m}^t \setminus \mathfrak{m}^{t+1}$ . Let  $x$  be a degree one superficial element of  $\mathcal{O}_C$ . Then  $x$  defines a non-zero divisor of  $\text{gr}_{\mathfrak{m}}(\mathcal{O}_C)$ . This implies that  $\mathfrak{m}^t / \mathfrak{m}^{t+1} \xrightarrow{x^{e_0-t}} \mathfrak{m}^{e_0} / \mathfrak{m}^{e_0+1}$  is a monomorphism. Notice that  $e_0 - t \geq e_0 + 1 - v^i$ , so

$h\bar{x}^{e_0-t} \in \mathfrak{m}^{e_0+1}$ . Since the coset of  $h$  in  $\mathfrak{m}^t/\mathfrak{m}^{t+1}$  is non-zero we get a contradiction. Hence  $h \in \mathfrak{m}^{v^i}$ , and we get the result.

(2) We only need to prove that  $s = \binom{b-1+r}{r+1}$  and  $v^i(I(C)) = r + 1$  for  $i = 1, \dots, s$ . This follows from [36].  $\square$

Notice that very often if  $p$  is a rigid Hilbert polynomial then the associated graded ring is Cohen-Macaulay, see for instance [15], [13].

**Corollary 4.7.** *Let  $x$  be a closed point of  $\mathbf{H}_{2,p(T)}$ . Then there exists a natural bijection between  $T_x$  and  $\mathfrak{m}_x^{e_0}$ .*

We will end this paper studying the local structure of the closed points of  $\mathbf{H}_{N,p}$ .

**Definition 4.8.** *Let  $C$  be a curve singularity with Hilbert polynomial  $p(T)$ . We say that  $C$ , or the corresponding closed point  $x$  of  $\mathbf{H}_{N,p}$ , is non-obstructed if the functor  $\underline{H}_{N,p(T)}^x$  is smooth, i.e. for all epimorphism of local finitely generated Artinian  $\mathbf{k}$ -algebras  $h : A \rightarrow A'$  the set map*

$$\underline{H}_{N,p(T)}^x(\mathrm{Spec}(A)) \xrightarrow{\underline{H}_{N,p(T)}^x(h)} \underline{H}_{N,p(T)}^x(\mathrm{Spec}(A'))$$

*is surjective.*

Let  $\mathbf{k} - \mathbf{mod}$  be the category of  $\mathbf{k}$ -modules. Given a  $\mathbf{k}$ -module  $W$  we will denote by  $W^* = \mathrm{Hom}_{\mathbf{k}}(W, \mathbf{k})$  its dual space. It is well known that there exists a natural monomorphism  $W \rightarrow W^*$ , that it is isomorphism if and only if  $W$  is a finite dimensional  $\mathbf{k}$ -module.

Following Laudal, [28], pag. 102, we will consider each object of  $\mathbf{k} - \mathbf{mod}$  endowed with a topology that will induce its reflexivity. Let  $W$  be an object of  $\mathbf{k} - \mathbf{mod}$ , pick a  $\mathbf{k}$ -base  $\mathcal{V} = \{v_i\}_{i \in I}$ . We put on  $W$  the topology for which a basis of neighborhoods of the neutral elements consists of the subspaces of  $W$  containing all but a finite number of the elements of  $\mathcal{V}$ . We will denote by  $\mathbf{k} - \mathbf{top.mod}$  the corresponding category of topological  $\mathbf{k}$ -modules. Given an object of  $\mathbf{k} - \mathbf{top.mod}$ . We will denote by  $W^\circ$  the topological dual of  $W$ . It is easy to see that the dual basis of  $\mathcal{V}$  defines a topology on  $W^\circ$  such that  $W \cong W^{\circ\circ}$ .

We know that for all closed point  $x$  of  $\mathbf{H}_{N,p}$  it holds

$$\mathcal{O}_{\mathbf{H}_{N,p},x} \cong \varinjlim_n \pi_n^*(\mathcal{O}_{\Xi_n, x_n}),$$

where  $x_n = \pi_n(x)$ , [21], 8.2.12.1. In particular we have  $\mathbf{n}_x \cong \varinjlim_n \pi_n^*(\mathbf{n}_{x_n})$ , where  $\mathbf{n}_{x_n}$  is the maximal ideal of  $\mathcal{O}_{\Xi_n, x_n}$ . For all  $n \geq e_0 + 1$  we pick a  $\mathbf{k}$ -basis  $\mathcal{V}_n = \{e_j\}_{j \in \mathcal{J}_n}$  of  $\mathbf{n}_{x_n}$ , such that  $\mathcal{J}_n \subset \mathcal{J}_{n+1}$ , and  $\mathcal{V}_n \subset \mathcal{V}_{n+1}$ . Notice that  $\mathcal{V} = \varinjlim_n \mathcal{V}_n$  is a  $\mathbf{k}$ -basis of  $\mathbf{n}_x$ , and  $\mathcal{V} \cup \{1\}$  is a  $\mathbf{k}$ -basis of  $\mathcal{O}_{\mathbf{H}_{N,p},x}$ . We write  $\mathcal{J} = \varinjlim_n \mathcal{J}_n$ . From now on we will consider  $\mathcal{O}_{\mathbf{H}_{N,p},x}$  endowed with the topology for which a basis of neighborhoods of the neutral element are the ideals containing all but a finite number of elements of  $\mathcal{V}$ . This topology will permit us to characterize the non-obstructiveness of the closed points of  $\mathbf{H}_{N,p}$ , see Theorem 4.9. We will denote by  $\mathcal{O}_{\mathbf{H}_{N,p},x}^+$  its completion with respect the topology defined above. This topology

induces a topology on  $\mathbf{n}_x/\mathbf{n}_x^2$ , making this  $\mathbf{k}$ -module an object of  $\mathbf{k} - \mathbf{top.mod}$ . Moreover, this topology induces also a topology on the tangent space  $T_x$ , in which the neighborhoods of the neutral element of  $T_x$  are the linear maps  $\omega : \mathbf{n}_x/\mathbf{n}_x^2 \longrightarrow \mathbf{k}$  whose kernel contains all but a finite number of elements of  $\mathcal{V}$ .

Let us consider the  $\mathbf{k}$ -algebra morphism  $\varphi : \mathbf{k}[T_i, i \in \mathcal{V}] \longrightarrow \mathcal{O}_{\mathbf{H}_{N,p},x}$  such that  $\varphi(T_i) = e_i$ ,  $i \in \mathcal{J}$ . Notice that  $\varphi_2 : \mathbf{k}[T_i, i \in \mathcal{V}]/(T_i)^2 \longrightarrow \mathcal{O}_{\mathbf{H}_{N,p},x}/\mathbf{n}_x^2$  is an isomorphism of  $\mathbf{k}$ -modules. We will consider in  $\mathbf{k}[T_i, i \in \mathcal{V}]$  the topology for which a basis of the neutral element are the ideals  $I$  contained in  $(T_i)$  such that all but a finite number of  $T_i$  belongs to  $I$ . We will denote by  $\mathbf{k}[T_i, i \in \mathcal{V}]^+$  the completion of  $\mathbf{k}[T_i, i \in \mathcal{V}]$  with respect this topology. Notice that  $\varphi$  is a continuous map with respect to the topologies defined above.

**Theorem 4.9.** *A closed point  $x$  of  $\mathbf{H}_{N,p}$  is a non-obstructed point if and only if*

$$\varphi^+ : \mathbf{k}[T_i; i \in I]^+ \longrightarrow \mathcal{O}_{\mathbf{H}_{N,p},x}^+$$

*is an isomorphism of  $\mathbf{k}$ -algebras.*

*Proof.* We will write  $S = \mathbf{k}[T_i, i \in \mathcal{V}]$ , and  $\mathcal{O}_x = \mathcal{O}_{\mathbf{H}_{N,p},x}$ . We will denote by  $I_n$  the ideal of  $S$  generated for  $T_i$ ,  $i \in \mathcal{J} \setminus \mathcal{J}_n$ ; and we will denote by  $W_n$  the corresponding ideal of  $\mathcal{O}_x$ . Hence  $\varphi$  induces an epimorphism of  $\mathbf{k}$ -vector spaces  $\varphi_n : S/I_n \longrightarrow \mathcal{O}_x/W_n$ . Since the set of  $\{I_n\}_n$ , resp.  $W_n$ , are cofinal in the basis of neighborhoods of  $S$ , resp.  $\mathcal{O}_x$ , we get that  $\varphi^+$  is an isomorphism of  $\mathbf{k}$ -algebras if and only if  $\varphi_n$  is an isomorphism for a  $n$  big enough.

Let us assume that  $x$  is non-obstructed and that  $\varphi_n$  is not an isomorphism; let  $F$  be a non-zero element of  $S/I_n$  belonging to the Kernel of  $\varphi_n$ . Let  $s$  be an integer such that the coset of  $F$  in  $A' = S/I_n + (T_i)^s$  is non-zero. Let  $h$  be morphism induced by  $\varphi_n$

$$h : A' = S/I_n + (T_i)^s \longrightarrow A = \mathcal{O}_x/W_n + \mathbf{n}_x^s.$$

Let us consider the projection  $\pi : \mathcal{O}_{\mathbf{H}_{N,p},x} \longrightarrow A$ . Since  $x$  is a non-obstructed point we get that there exists a morphism  $\sigma : \mathcal{O}_{\mathbf{H}_{N,p},x} \longrightarrow A'$  such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{O}_{\mathbf{H}_{N,p},x} & & \\ \sigma \downarrow & \searrow \pi & \\ A' & \xrightarrow{h} & A \end{array}$$

Since  $\varphi_2$  is an isomorphism we get that the above diagram induces

$$\begin{array}{ccc} A & & \\ \bar{\sigma} \downarrow & \searrow \bar{\pi}=Id & \\ A' & \xrightarrow{h} & A \end{array}$$

so  $h$  is an isomorphism and we get a contradiction.

If  $\varphi^+$  is an isomorphism then it is easy to see that  $x$  is non-obstructed. □

**Proposition 4.10.** (i) Every closed point  $x$  of  $\mathbf{H}_{2,p(T)}$ , with  $p(T) = e_0T - e_0(e_0 - 1)/2$ , is non-obstructed. In particular  $\mathbf{H}_{2,p(T)}$  is reduced.

(ii) Given an integer  $a \geq 2$  we put  $p_a = \binom{a}{2}T - 2\binom{a}{3}$ . Then every closed point  $x$  of  $\mathbf{H}_{3,p_a}$  is non-obstructed, and  $\mathbf{H}_{3,p_a}$  is reduced.

*Proof.* (i) Let  $h : A \rightarrow A'$  be a epimorphism of local finitely generated local  $\mathbf{k}$ -algebras; we need to prove that  $\underline{H}_{2,p(T)}^x(h)$  is surjective. We write  $S = \text{Spec}(A)$  and  $S' = \text{Spec}(A')$ .

Given a family  $Z' \in \underline{H}_{2,p(T)}^x(S)$  there exists  $F \in A'[[X_1, X_2]]$  such that  $Z' = \text{Spec}(A'[[X_1, X_2]]/(F))$ . Let  $G \in A[[X_1, X_2]]$  be a power series of order  $e$  and such that  $h(G) = F$ . Then it is easy to prove that the family  $Z = \text{Spec}(A[[X_1, X_2]]/(G))$  verifies  $\underline{H}_{2,p(T)}^x(h)(Z) = Z'$ .

(ii) Let  $h : A \rightarrow A'$  be a epimorphism of local finitely generated local  $\mathbf{k}$ -algebras; we write  $S = \text{Spec}(A)$  and  $S' = \text{Spec}(A')$ . Let  $Z'$  be a family of curve singularities of  $(\mathbf{k}^3, 0)$  over  $S'$  with Hilbert polynomial

$$p_a(T) = \binom{a}{2}T - 2\binom{a}{3}.$$

Let  $x$  be the closed point of  $\mathbf{H}_{3,p_a(T)}$  defined by the closed fiber of  $Z'$ . From Corollary 4.5, we get that  $v(I_x) = a$  and  $I_x \subset R = \mathbf{k}[[X_1, X_2, X_3]]$  admits a minimal free resolution

$$0 \rightarrow R^{a-1} \xrightarrow{M} R^a \rightarrow I_x \rightarrow 0$$

with  $M$  a matrix with entries belonging to the maximal ideal of  $R$  of order 1. Hence there exists an  $a \times (a-1)$  matrix  $\overline{M}$  with coefficients in the maximal ideals of  $A[[X_1, X_2, X_3]]$ , such that its maximal minors generates an ideal, say  $I'$ , with  $Z' = \text{Spec}(A'[[X_1, X_2, X_3]]/I')$  and  $\overline{M}$  is mapped to  $M$  by the natural morphism of  $\mathbf{k}$ -algebras  $A'[[X_1, X_2, X_3]] \rightarrow \mathbf{k}[[X_1, X_2, X_3]]$ . Let  $N$  be an  $a \times (a-1)$  matrix such that its entries belong to the maximal ideal of  $A$  and are mapped to the entries of  $\overline{M}$  by  $h$ . Let  $I$  be the ideal of  $A[[X_1, X_2, X_3]]$  generated by the maximal minors of  $N$ ; we put  $Z = \text{Spec}(A[[X_1, X_2, X_3]]/I)$ . It is easy to see that  $Z \in \underline{H}_{3,p_a(T)}^x(S)$ , where  $S = \text{Spec}(A)$ , and  $Z' = \underline{H}_{3,p_a(T)}^x(h)(Z)$ .  $\square$

## REFERENCES

- [1] M. Artin. Algebraic approximation of structures over complete local rings. *Publ. I.H.E.S.*, 36:23–58, 1969.
- [2] M. Artin. Deformations of singularities. *Tata Inst. Lecture Notes*, 1976.
- [3] V. Batyrev. Stringy Hodge numbers of varieties with Gorenstein canonical singularities. In *Integrable systems and algebraic geometry, Kobe/Kyoto, 1997*, pages 1–32, 1998.
- [4] J. Briançon and A. Iarrobino. Dimension of the punctual Hilbert scheme. *J. of Algebra*, 55(536-544):536–544, 1978.
- [5] R.O. Buchweitz and G. M. Greuel. The Milnor number and deformations of complex curve singularities. *Inv. Math.*, 58:241–281, 1980.
- [6] J. Denef and F. Loeser. Germs of arcs on singular algebraic varieties and motivic integration. *Invent. Math.*, 135(1):201–232, 1999.
- [7] J. Denef and F. Loeser. Geometry on arc spaces of algebraic varieties. In *Proceedings of the 3 European Congress of Mathematics*, volume 201 of *Progr. Math.*, pages 327–348. Birkhauser, 2000.
- [8] J. Denef and F. Loeser. Motivic integration, quotient singularities and McKay correspondence. *Compositio Math.*, 131:267–290, 2002.

- [9] J.A. Eagon and D.G. Northcott. Ideals defined by matrices and a certain complex associated with them. *Proc. Royal Soc. England*, A269:188–204, 1962.
- [10] J. Elias. A sharp bound for the minimal number of generators of perfect height two ideals. *Manuscripta Math.*, 55:93–99, 1986.
- [11] J. Elias. Characterization of the Hilbert-Samuel polynomials of curve singularities. *Compositio Math.*, 74:135–155, 1990.
- [12] J. Elias. The conjecture of Sally on the Hilbert function for curve singularities. *J. Algebra*, 160(1):42–49, 1993.
- [13] J. Elias. On the depth of the tangent cone and the growth of the Hilbert function. *Trans. Amer. Math. Soc.*, 351(10):4027–4042, 1999.
- [14] J. Elias. On the deep structure of the blowing-up of curve singularities. *Math. Proc. Cambridge Philos. Soc.*, 131:227–240, 2001.
- [15] J. Elias and G. Valla. Rigid Hilbert functions. *J. of Pure and App. Algebra.*, 71:19–41, 1991.
- [16] R. Gilmer. On polynomials over a Hilbert ring. *Michigan Mayh. J.*, 18:205–212, 1971.
- [17] Gert-Martin Greuel and Gerhard Pfister. Moduli for singularities. In Cambridge Univ. Press, editor, *Singularities (Lille, 1991)*, volume 201 of *London Math. Soc. Lecture Note Ser.*, pages 119–146. Cambridge Univ. Press, 1994.
- [18] A Grothendieck. Les schémas de Hilbert. *Sem. Bourbaki*, 220, 1960.
- [19] A. Grothendieck, P. Berthelot, and L. Illusie. Théorie des Topos et Cohomologie Etale de Schémas. *L.N.M.*, 269, 270, 305, (1972-73).
- [20] A. Grothendieck and J. Dieudonné. Etude globale élémentaire de quelques classes de morphismes. *Publ. Math. IHES*, 8, 1961.
- [21] A. Grothendieck and J. Dieudonné. Etude locale des schémas et des morphismes de schémas. *Pub. Math. IHES*, 20, 24, 28, 32, (1964-67).
- [22] A. Grothendieck and J. Dieudonné. Eléments de Géometrie Algébrique, I. *Grundlehren, Springer-Verlag*, 166, 1971.
- [23] G. Hermann. Die frage der endlich vielen schritte in der theorie der plynomideale. *Math. Ann.*, 95:726–788, 1926.
- [24] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. *Annals of Math.*, 79(2):109–326, 1964.
- [25] D. Kirby. The defect of a one-dimensional local ring. *Mathematika*, 6:91–97, 1959.
- [26] D. Kirby. The reduction number of a one-dimensional local ring. *J. London Math. Soc.*, 10:471–481, 1975.
- [27] M. Kontsevich. Motivic integration. *Lecture at Orsay*, 7 Dec 1995.
- [28] O.A. Laudal. *Formal Moduli of Algebraic Structures*, volume 754 of *L.N.M.* 1979.
- [29] O.A. Laudal and G. Pfister. *The Local Moduli Problem. Applications to Isolated Hypersurface Singularities.*, volume 1310 of *L.N.M.* Springer Verlag, 1988.
- [30] E. Looijenga. Motivic Measures. *Astérisque ( Séminaire Bourbaki, Vol. 1999/2000, Exposé 874)*, 276:267–297, 2002.
- [31] E. Matlis. 1-dimensional Cohen-Macaulay rings. *L.N.M. Springer Verlag*, 327, 1977.
- [32] H. Matsumura. *Commutative Algebra*. Benjamin, second edition, 1980.
- [33] D. Mumford and J. Fogarty. Geometric invariant theory, 2nd edition. *Ergebnisse, Springer-Verlag*, 1982.
- [34] P.E. Newstead. Introduction to moduli problems and orbit spaces. *Tata Inst. Lecture Notes*, 1978.
- [35] D.G. Northcott. On the notion of the first neighbourhood ring with application to the  $a\mathfrak{f} + b\varphi$  theorem. *Proc. Camb. Phil. Soc.*, 53:43–56, 1957.
- [36] F. Orecchia. Maximal Hilbert functions of one-dimensional local rings. *L.N. in Pure and Appl. Math. Series, Marcel Dekker*, 84, 1983.
- [37] L. Robbiano and Valla G. On the equations defining tangent cones. *Math. Proc. Camb. Phil. Soc.*, 88, 1980.
- [38] J. Sally. Bounds for the number of generators of Cohen-Macaulay ideals. *Pacific J. of Math.*, 63:517–520, 1976.
- [39] R. Sjögren. On the regularity of graded  $k$ -algebras of Krull dimension  $\leq 1$ . *Math. Scan.*, 2:167–172, 1992.
- [40] R.P. Stanley. Hilbert functions of graded algebras. *Adv. Math.*, 28:57–83, 1978.



- [41] B. Teissier. The hunting of invariants in the geometry of the discriminant. *Real and complex singularities, Sitjthoff and Noordhoff*, 1977.
- [42] O. Zariski. Le probleme des modules pour les branches planes. *Publ. de Centre de Math. de L'Ecole Polytechnique de Paris*, 1973.
- [43] O. Zariski and P. Samuel. *Commutative Algebra II*, volume 29 of *Graduate Texts in Mathematics*. Springer Verlag, 1975.

DEPARTAMENT D'ALGEBRA I GEOMETRIA  
FACULTAT DE MATEMÀTIQUES  
UNIVERSITAT DE BARCELONA  
GRAN VIA 585, 08007 BARCELONA, SPAIN  
*E-mail address:* elias@ub.edu